

Credit Risk and Sustainable Debt: A Model and Estimations for Euroland*

Willi Semmler[†] and Malte Sieveking[‡]

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Abstract

The paper studies credit risk and sustainable debt in the context of an intertemporal model. For a dynamic growth model with an additional equation for the evolution of debt we demonstrate that sustainable debt may typically be state constrained. In order to control credit risk the lender needs to know the sustainable debt of the borrower at each point in time. We compute sustainable debt by applying new methods. Using those new methods we can demonstrate the region in which there is no credit risk and where the borrower remains creditworthy. Even for state dependent credit cost we can study the debt capacity of a borrower and the role of debt ceilings for borrowing and lending. We discuss continuous and discrete time variants of optimal growth models with borrowing and lending and provide error bounds for the discretized version. The analytics is provided for a general model and some generic results are presented for a univariate problem. Our new methods also permits us to study the problem of multiple steady states which arise from nonlinear adjustment cost of capital. We can determine optimal and non-optimal solutions and cut-off points where domains of attraction separate to high and low level equilibria. The dynamic model is then estimated by employing aggregate data for the core countries of Euroland. Moreover, the sustainability of external debt is also estimated for those core countries of Euroland. Those estimations are undertaken in order to spell out implications for the stability of the Euro.

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[†]Department of Economics, University of Bielefeld, 33501 Bielefeld, Germany, New School University, 65 Fifth Ave, New York, NY 10003, e-mail:wsemmler@wiwi.uni-bielefeld.de and semmlerw@newschool.edu

[‡]Department of Mathematics, University of Frankfurt, Robert Meyer Str. 7-11, Frankfurt, Germany.

1 Introduction

Recently, asset markets have been studied in the context of stochastic growth models in numerous papers. The particular focus was, however, mostly on the stock market and whether characteristics of the stock market such as the excess volatility (Danthine, Donaldson and Mehra 1992), the excess of stock returns over the risk-free rate or the Sharpe-ratio can be replicated in the context of stochastic growth models (see Lettau 1999, Lettau and Uhlig 1999, Lettau, Gong and Semmler 1997 and Woehrmann, Semmler and Lettau 1999). Credit market issues such as the returns from bonds, in particular short and long bonds and the premium they carry over the risk-free rate have also been studied in the context of those models (see Lettau 1999 and Lettau and Uhlig 1999). This paper focuses less on returns and spreads arising in asset markets but rather wants to study and evaluate credit risk in the context of a dynamic economic model. More specifically we want to study borrowing capacity, creditworthiness and credit risk in the context of an economic growth model. In order to simplify matters we do not employ a stochastic version of a growth model but rather employ a deterministic framework.¹ Yet, our study might still be important for issues of credit risk and for risk management that recently have been discussed in many empirical contributions.² Although our study has implications for credit risk analysis in empirical finance literature our paper here is more specifically related to the literature that link credit market and economic activity in the context of intertemporal models. In recent times this link has been explored in numerous papers.

In a first type of papers, mostly assuming perfect credit markets, it is assumed that, roughly speaking, agents can borrow against future income as long as the discounted future income, the wealth of the agents, is no smaller than the debt that agents have incurred. In this case there is no credit risk whenever the non-explosiveness condition holds. Positing that the agents can borrow against future income the non-explosiveness condition is equivalent to the requirement that the intertemporal budget constraint holds for the agents. Formally, the necessary conditions for optimality, derived from the Hamiltonian equation, are often employed to derive the dynamics of the state variables and the so called transversality condition is used to provide a statement on the non-explosiveness of the debt of the economic agents. Models of this type have been discussed in the literature for households, firms, governments and small open economies (with access to international capital markets) where the transversality condition is employed to disregard the evolution of debt.³

In a second type of papers, and also often in practice, assuming credit market imperfec-

¹A stochastic version can be found in Sieveking and Semmler (1999).

²For empirical methods to evaluate credit risk and to compute default adjusted bond rates, see Beninga (1998, ch. 17). Those methods are very useful in practice but have only little connection to a theory of credit risk. Another approach in risk management is the value at risk approach working with expected volatility of asset prices, for a survey see Duffie and Pan (1997)

³For a brief survey of such models for households', firms' and governments or countries, see Blanchard and Fischer (1989, ch.2) and Turnovsky (1995).

tions, economists presume that borrowing is constrained. Frequently, borrowing ceilings are assumed which are supposed to prevent agents from borrowing an unlimited amount. Presuming that agents' assets serve as collateral a convenient way to define the debt ceiling is then to assume the debt ceiling to be a fraction of the agents' wealth. The definition of debt ceilings have become standard, for example, in a Ramsey model of the firm, see Brock and Dechert (1985) or in a Ramsey growth model for small open economies; see, for example, Cohen and Sachs (1986) and Barro, Mankiw and Sala-i-Martin (1995). It has also been pointed out that banks often define debt ceilings for their borrowers, see Bhandari, Haque and Turnovsky (1990).

A third type of literature also assumes credit market imperfections but employs endogenous borrowing cost such as in the work by Bernanke and Gertler (1989, 1994) and further extensions to heterogenous firms, such as small and large firms, in Gertler and Gilchrist (1994). Often here one presupposes only a one period zero horizon model and then it is shown that due to endogenous change of net worth of firms, as collateral for borrowing, credit cost is endogenous. For potential borrowers their credit cost is inversely related to their net worth. In parallel other literature has posited that borrowers may face a risk dependent interest rate which is assumed to be composed by a market interest rate (for example, an international interest rate) and an idiosyncratic component determined by the individual degree of risk of the borrower. Various forms of the agent specific risk premium can be assumed. Here, often it is posited to be endogenous in the sense that it is convex in the agents' debt.⁴

Recent extensions of the third type of work have been undertaken by embedding credit market imperfections and endogenous borrowing cost more formally in intertemporal models such as the standard stochastic growth model, see Carlstrom and Fuerst (1997), Bernanke, Gertler and Gilchrist (1998), Cooley and Quadrini (1998) and Krieger (1999). Some of this literature has dealt also with borrowing constraints of heterogenous agents (households, firms) in a intertemporal general equilibrium framework. Although in our paper we stress intertemporal behavior of economic agents in the context of a growth model, we here will not deal with the case of heterogenous agents.

With respect to the first type of papers, this article demonstrates that the borrowing capacity is state constrained and shows the regions where debt is feasible and the borrower remains creditworthy. Below this ceiling there is no credit risk. We demonstrate that the debt ceiling should not be arbitrarily defined. We can compute the debt capacity which in our analysis will be defined by a curve of sustainable debt even if the interest rate is a function of the state variables. Of course, in practice insolvency of the borrower can arise without the borrower moving up to his/her borrowing capacity. Insolvency may occur when a borrower faces a loss of his/her "reputational collateral" (Bulow and Rogoff 1989) without having reached the debt capacity. A country, for example, losing its creditworthiness may then face a sudden reversal of capital flows giving rise to a currency and financial crises and large output loss.⁵ We want to stress, however, that in our paper

⁴The interest rate as function of the default risk of the borrower is posited by Bhandari, Haque and Turnovsky (1990), Rauscher (1990) and Turnovsky (1995).

⁵Such reversal of capital flows have recently been studied empirically in a series of papers by Milesi-

we are concerned with the "ability to pay" and less with the borrower's "willingness to pay".⁶

We also show that fixing debt ceilings, as in the second type of papers, may be arbitrary and lead to welfare losses.⁷ As we will show it might often be useful first to expand debt before it can be reduced. We want to argue against the usual use of 'ceilings' if they differ from what we define as creditworthiness. Either the ceiling is too high and the debtor might be tempted to move close to the ceiling and then goes bankrupt or the ceiling is too low, then the agent may not be able to develop its full potentials, and thus face a welfare loss, or it may be the case that the contract is not feasible whereas it would be feasible if the debt ceiling is higher.

Lastly, as in the third type of papers, we compute endogenous borrowing constraints by computing maximum debt capacity of an economic agent by making the credit cost state dependent. There is then also at each point in time a constraint for the maximum amount the agent can borrow.⁸ Our analysis therefore also stresses the role of balance sheets for borrowing as in the third approach but the (maximum) liabilities are derived from the present value of net income of agents. Yet, as we will show, the credit cost, instead of being determined by net worth, is affected negatively by assets (collateralized capital stock) and positively impact by existing debt which make the credit cost endogenous. As we will show, however, in particular this problem makes the standard present value approach difficult to apply.

Although our methods we use here to study credit market and economic activity are very generic in the main we restrict our study to an example of an open economy that borrows from international capital markets. Yet, our analysis shows implications for a wider range of intertemporal models with borrowing and lending. In our paper (in its general version) we posit that the country is endowed with a resource⁹ and extracts and/or accumulates capital to generate income and to service the debt. The current account is determined by intertemporal decisions to invest and to consume. In studying borrowing and lending in the context of a growth model we show that multiple steady states may arise. In our model this comes from the fact that there is a nonlinear adjustment cost for the capital stock.

By undertaking our study we attempt to bypass utility theory. Economists have argued that analytical results in intertemporal models often depend on the form of the utility function employed.¹⁰ Moreover, so it is argued, economic theory should not necessarily be

Ferretti and Razin (1996, 1997). Their latest paper studies the Asian financial crisis in the light of the above approach to insolvency.

⁶Recent development on the latter type of literature, in particular on the problem of incentive compatible contracts is surveyed in Eaton and Fernandez (1995).

⁷A more elaborate analysis of how credit ceilings may lead to welfare losses is given in Semmler and Sieveking (1996).

⁸We thus do not have to introduce the restriction of a one period contract as in Bernanke, Gertler and Gilchrist (1998).

⁹A more specific model referring to renewable resources instead of capital stock as a state equation can be found in Semmler and Sieveking (1999).

¹⁰A related model, however, working with a utility functional can be found in Semmler and Sieveking

founded on the notion of utility since such a foundation is not well supported by empirical analysis.¹¹ We want to argue that some parts of the theory, such as creditworthiness, can be formulated without the use of utility theory.¹²

We employ new analytical and numerical techniques to study the proposed model. One of the major new techniques we use is vector field analysis which allows to study intertemporal problems with state constraints. This technique permits to study the direction of the vector field in the state space resulting from a certain control actions. This helps to answer the question whether a control will be able to steer the debt bounded. The method permits also to compute a critical curve below which the debt is sustainable – the debt can be steered bounded– and above which the debt is explosive. We also demonstrate the relation of vector field analysis to the Hamilton-Jacobi-Bellman (H.J.B.) equation used in dynamic programming. We also show of how the Hamiltonian equation derived from Pontryagin’s maximum principle can be used to study the problem of credit risk and creditworthiness.

A particular problem that arises in the context of models of multiple steady states is to show which steady states are optimal and which not. Our new methods that we propose admit to study multiple equilibria and it helps to distinguish between optimal and non-optimal solutions. We compute the global dynamics numerically and can locate cut-off points in the sense of Skiba (1978) where domains of attraction separate to high and low level equilibria.¹³ Our proposed methods can numerically determine those cut-off points the location of which will depend on borrowing cost.

Dynamic economic models are usually stated in either discrete time or continuous time form. In case of a continuous time form the numerical treatment, and thus the computation of the critical curve below the debt is sustainable, requires to transform the model into a discrete time form. We thus need to show how we can discretized the continuous time equations. We employ the Euler approximation and give estimates of the error bound. The treatment of this problems as well as generalizations to the multivariate case is discussed in appendices. As aforementioned the literature often suggests that for a theory of debt contracts one should use a stochastic model since lending involves risk. A stochastic analysis - which is a preferable approach to pursue - leads to Hamilton-Jacobi-Bellman equations with additional state variables, namely with stochastic ones. In the present article we first ignore stochasticity because our problem whether or not a contract is feasible for an initial debt seems easier to treat without uncertainty and at the same time this might serve a useful exercise for the analysis of the stochastic case. A stochastic version is provided in Sieveking and Semmler (1999).¹⁴

Finally, we want to point out that the model we are proposing and analytically and

(1999). In fact, our methods also allows to compute welfare out of the steady state.

¹¹In particular Hildenbrand and Kneip (1997) have recently argued that economic theory should refrain from unobservables and employ observable variables as much as possible.

¹²An analytical treatment why and under what conditions the creditworthiness problem can be separated from the consumption problem is given in Semmler and Sieveking (1996).

¹³For an extensive survey of the earlier work on Skiba-points, see Brock and Malliaris (1996), ch. 6.

¹⁴In the latter paper we question the view that in the stochastic case one can simply replace the present value computation by the computation of the expected present value.

numerically study can be empirically applied. We take our model to the data by transforming our model into an estimable form. A natural candidate to apply our proposed model would be the countries that triggered the financial crisis in Asia.¹⁵ Since the data problems there are, however, quite severe we apply our model to Euroland and estimate the parameters by employing aggregate time series data for the core countries of Euroland for the period 1978 to 1998. This helps us to evaluate whether external debt of Euroland is sustainable and to evaluate the external strength of the Euro.

2 A Brief Overview

Next, we want to give a more formal overview on our results. In a contract between a creditor and debtor there are two measurement problems involved. The first pertains to the computation of debt and the second to the computation of the debt ceiling. The first problem is usually answered by employing an equation of the form

$$\dot{B}(t) = \theta B(t) - f(t), \quad B(0) = B \quad (1)$$

where $B(t)$ is the level of debt¹⁶ at time t , θ the credit cost and $f(t)$ the debt service. The second problem can be settled by defining a debt ceiling such as

$$B(t) \leq C, \quad (t > 0)$$

or less restrictively by

$$\sup_{t \geq 0} B(t) < \infty$$

or even less restrictively by the aforementioned transversality condition

$$\lim_{t \rightarrow \infty} e^{-\theta t} B(t) = 0.$$

The ability of a debtor to service the debt, i.e. the feasibility of a contract, will depend on the debtors source of income. Along the line of intertemporal models with borrowing and lending¹⁷ we model this source of income as arising from a stock of capital $k(t)$, at time t , which changes with investment rate $j(t)$ at time t through

$$\dot{k}(t) = j(t) - \sigma(k(t)), \quad k(0) = k. \quad (2)$$

¹⁵For recent interpretation of the Asian financial crisis making the employing the argument of the balance sheets central, see Mishkin (1998), Krugman (1999) and Burnside, Eichenbaum and Rebelo (1999).

¹⁶Note that all subsequent state variables are written in terms of efficiency labor along the line of Blanchard (1983), for details see section 4.

¹⁷Prototype models used as basis for our further presentation can be found in Blanchard (1983), Blanchard and Fischer (1989) or Turnovsky (1995).

In our general model both the capital stock and the investment are allowed to be multivariate. As debt service we take the net income rate from the investment rate $j(t)$ at capital stock level $k(t)$ minus some minimal rate of consumption.¹⁸ Hence

$$\dot{B}(t) = H(k(t), B(t)) - f(k(t), j(t)), \quad B(0) = B \quad (3)$$

where $H(k(t), B(t))$ is the credit cost. Note that the credit cost is not necessarily a constant factor (a constant interest rate). As debt ceiling we take

$$\sup_{t \geq 0} B(t) < \infty \quad (4)$$

Let us call an initial indebtedness B subcritical for an initial capital stock k if there is an investment function $j(\cdot)$ such that the corresponding solution $t \rightarrow (B(t), k(t))$ of (2), (3) satisfies (4). Let $B^*(k)$ be the supremum of all initial levels of debt which are subcritical for initial capital k . We call $B^*(k)$ the creditworthiness of the capital stock k . The problem to be solved in this paper is how to compute B^* .

If there is a constant credit cost factor (interest rate), $\theta = \frac{H(B,k)}{B}$, then as is easy to see $B^*(k)$ is the present value of k :

$$B^*(k) = \underset{j}{Max} \int_0^{\infty} e^{-\theta t} f(k(t), j(t)) dt - B(0) \quad (5)$$

$$s.t. \quad \dot{k}(t) = G(k(t), j(t)), \quad t \geq 0, \quad k = k(0). \quad (6)$$

The more general case is, however, that θ is not a constant. As in the theory of credit market imperfections we generically may let θ depend on k and B .¹⁹ Then not only the relation of the present value to creditworthiness but also the notion of present value itself become difficult to treat.

Note that in the model (5)-(6) we have not used utility theory. As shown in Sieveking and Semmler (1998) the model (5)-(6) exhibits, however, a strict relationship to a growth model built on a utility functional, for example, such as²⁰

$$Max \int_0^{\infty} e^{-\theta t} u(c(t), k(t)) dt \quad (7)$$

$$s.t. \quad \dot{k}(t) = G(k(t), j), \quad k(0) = k. \quad (8)$$

¹⁸In the subsequent analysis of creditworthiness we can set consumption equal to zero. Any positive consumption will move down the creditworthiness curve. Note also that public debt for which the Ricardian equivalence theorem holds, i.e. where debt is serviced by a non-distortionary tax, would cause the creditworthiness curve to shift down. In computing the "present value" of the future net surpluses we do not have to assume a particular interest rate. Yet, in the following study we neither elaborate on the problem of the price level nor on the exchange rate and its effect on net debt and creditworthiness.

¹⁹Note again, that instead of relating the credit cost inversely to net worth, as in Bernanke, Gertler and Gilchrist (1998), we use the two arguments, k and B , explicitly.

²⁰For details, see Blanchard (1983).

$$\dot{B}(t) = \theta B(t) - f(k(t), j) + c(t), \quad B(0) = B \quad (9)$$

with the transversality condition

$$\lim_{t \rightarrow \infty} e^{-\theta t} B(t) = 0 \quad (10)$$

which often turns up in the literature²¹ among the "necessary conditions" for a solution of a welfare problem such as (7)-(9).

In Sieveking and Semmler (1998) it is shown that the problem (7)-(9) can be separated into two problems. The first problem is to find optimal solutions that generate the present value of net income flows and the second problem is to study the path of how the present value of net come flows is consumed. There also conditions are discussed under which such separation is feasible. The separation into those two problems appear to be feasible as long as the evolution of debt does not appear in the objective function. If such separation is feasible we then only need to be concerned with the model (5)-(6).

In the context of the model (5)-(6) as well as in a generalized version where generically the credit cost, θ , may depend on B and k we want to argue against the usual use of 'ceilings' if they differ from creditworthiness $B^*(k)$. Suppose the 'ceiling' is of the form

$$B(t) \leq C \quad \text{for all } t$$

Either $C > B^*(k)$, then the ceiling is too high because the debtor might be tempted to move close to the ceiling and then goes bankrupt if $B > B^*(k)$. Or $C < B^*(k)$, then the economy may not be able to develop its full potentials, and thus face a welfare loss²², or it may be the case that the contract is not feasible whereas it would be feasible if

$$B(t) \leq C \quad \text{for all } t$$

would be replaced by

$$\limsup_{t \rightarrow \infty} B(t) \leq C.$$

On the other hand, the last condition obviously is of no practical use if we can not say when $B(t) \leq C$.

We allow negative investment rates $j < 0$, i.e. reversible investment for simplicity. $B^*(k)$ does not change by the requirement $j \geq 0$ for large $\|k\|$, in the multivariate version, and also whenever debt control requires to increase capital ($j = j_+$ see Lemma 1, in appendix II). If, however, $\|k\|$ is small and debt control requires to decrease capital, then the most effective investment becomes more complicated.

²¹See, for example, Bhandari, Hague and Turnovsky (1990). In our framework the equivalent transversality condition will be

$$\sup_{t \geq 0} B(t) < \infty$$

²²In Semmler and Sieveking (1996) the welfare gains from borrowing are computed.

The demonstration of the existence of a solution of a model such as (5)-(6) is, however, often neglected in economics. It frequently has been argued that existence is a non-problem since non-existence only indicates that the problem is ill-posed. We would like to remark, however, that we have to care about the problem, since non-existence for an initial value problem $(k(0), B(0))$ means bankruptcy for the borrower with economic consequences for the lender as well.

The existence problem of our model is more specifically studied in the appendices II and III where also link between time continuous and time discrete models is explored. The latter is very simple and lends itself to an iterative dynamic programming solution which serves also as approximation for the continuous time case. In appendix II we study a continuous time model for a multi-variable case with capital stocks $k = (k_1, \dots, k_n)$. Some components k_i of which may be interpreted as natural resources and some components of the investment vector $j = (j_1, \dots, j_n)$ may be interpreted as extraction rates. We state the Hamilton-Jacobi-Bellman (H.J.B.) equation for $B^*(k)$ and show that $B^*(k)$ is the limit of the discrete time approximation of appendix III. In an extended version of the paper²³ we also discuss a discretization error bound which is linear in step size and which is derived both for the discretization of a continuous solution of differential inclusion and for the converse problem where one starts with a discrete time solution and looks for a continuous one which is close to it.

The remainder of the paper, which is devoted solely to the univariate case of a growth model with credit market, is organized as follows. Section 3 introduces three methods of how to study the global dynamics in economies which may exhibit multiple candidates for steady states. In section 4 the growth model with credit market is explicitly introduced. The above methods are employed to study the global dynamics of the model. In section 5 for particular parameter values the algorithms are employed and the global dynamics studied. Section 6 then estimates the parameter set involved in the growth model by using aggregate data for capital stock and investment for the core countries of Euroland. Section 7 undertakes the test of sustainability of external debt for Euroland. Sections 8 provides the conclusions. Appendix I provides some derivations of the Hamiltonian used in the paper and the appendices II and III provides the aforementioned study of the multivariate continuous as well as discrete time versions of the model.

3 Methods to Study the Global Dynamics

Let us write²⁴ a standard form of an infinite horizon optimal control problem $P(a)$ with a one dimensional state space as

²³See Semmler and Sieveking (1998a).

²⁴For details on the methodology proposed in the subsequent section, see Semmler and Sieveking (1998b). Other recent papers that study the problem of multiple steady states and the thereby arising Skiba-points are Wirl and Feichtinger (1999), Santos (1999), Brock and Starret (1999) and Semmler and Greiner (1999), for a Ramsey pricing model with Skiba-points, see Brock and Deckert (1985).

$$\begin{aligned}
& \text{Max} \int_0^\infty e^{-\delta t} f_0(x(t), u(t)) dt \\
& \text{s.t. } \dot{x} = f(x(t), u(t)) \\
& \quad u(t) \in U \\
& \quad x(0) = a
\end{aligned} \tag{11}$$

where $f_0(x(t), u(t))$ is a return function and $f(x(t), u(t))$ a function describing the dynamics in the state space.

If (x, u) is a solution, i.e. if $x(\cdot)$ is optimal then as t tends to infinity $x(t)$ will converge toward an optimal steady state - except in case where there is a continuum of such optimal steady state. Let us call an attractor an optimal stationary state \mathbf{a} which is approached from above and from below by solutions

$$\begin{aligned}
& a = \lim_{t \rightarrow \infty} x(t) = \lim_{t \rightarrow \infty} y(t) \\
& x(t) < a < y(t), \quad (t > 0).
\end{aligned} \tag{12}$$

A constant solution a , i.e. an optimal stationary state a , is called repeller if there are optimal x, y such that

$$\begin{aligned}
& a = \lim_{t \rightarrow -\infty} x(t) = \lim_{t \rightarrow -\infty} y(t) \\
& x(t) < a < y(t), \quad (t > 0)
\end{aligned} \tag{13}$$

Our conjecture is that generically (that is except for a "small" set of degenerated problems) there are no optimal repellers in one dimensional optimal control problems in contrast to the situation for differential equations where repellers are admissible. Exception has also to be made for the origin in problems like the following growth model with creed market laid out in section 4 where the state space is $[0, +\infty)$. In fact if the zero capital stock is an attractor there is no chance for a country with a small amount of investment to take off.

If our conjecture is true then if there are multiple (optimal) steady states, there are attractors $a_1 < a_2$ with no optimal stationary state in between, a situation which is impossible for differential equations.

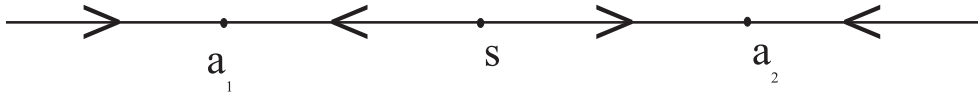


Figure 1:

The state most to the left of all states $a_1 < s < a_2$ which are optimally steered to a_2 is called a Skiba-point. The problem $P(s)$ will have two different solutions (x, u) and (y, u) respectively with

$$\lim_{t \rightarrow \infty} x(t) = a_1, \lim_{t \rightarrow \infty} y(t) = a_2, x(0) = y(0) = b \quad (14)$$

Note that such cut-off points do not exist for ordinary differential equations $\dot{x} = v(x)$ on the real line, if v is continuous. We demonstrate the existence of cut-off points below for a problem of optimal investment where investment cost becomes prohibitive for small capital stocks and where, beyond a certain threshold, the optimal investment increases the stock approaching an optimal stationary state.

Note that the (canonical) Hamiltonian equations may very well provide a candidate for a steady state b between a_1 and a_2 with admissible x, y such that

$$\lim_{t \rightarrow \infty} x(t) = a_1, \lim_{t \rightarrow \infty} y(t) = a_2, x(0) = y(0) = b \quad (15)$$

We maintain, however, that in such a case generically b is not optimal. Indeed, the stationary control u which keeps the state at b - if it exists - is not optimal. Intuitively, if x, b, y were optimal then a small increase in the pay-off f_0 for states b would increase the value of x and y but leave the value of b unchanged. Hence, b ceases to be optimal whereby we assume that by the small increase in f_0 no new repellors are created. The latter implies that the system is structurally stable. In order to detect optimal and non-optimal steady stages we can three different types of methods.

In the **first method** to solve $P(a)$ numerically we can use an algorithm as proposed by Semmler and Sieveking (1998a):²⁵

1. Solve the stationary Hamilton-Jacobi-Bellman (HJB)-equation

$$\begin{aligned} (HJB_0) \quad f_0(e) &= \text{Max} \left\{ f_0(e, u) + \frac{1}{\delta} f'_0(e) f(e, u) \mid u \in U \right\} \\ f_0(x) &= \text{Max} \{ f_0(x, u) \mid u \in U, f(x, u) = 0 \} \\ f'_0(x) &= \lim_{\substack{h \neq 0 \\ h \rightarrow 0}} \frac{1}{h} (f_0(x+h) - f_0(x)) \end{aligned} \quad (16)$$

for states e .

We want to remark that every optimal stationary state e solves (HJB_0) according to Sieveking (1988); the proof of this is elementary, it does not rely on the maximum principle but allows for a state dependent control set $U(x)$.

2. Solve the HJB-equation

$$(HJB) \quad \delta W(x) = \text{Max}_u \{ f_0(x, u) + W'(x) f(x, u) \}$$

to obtain $W'(x)$ explicitly as a function of x and $W(x)$ use

$$W'(x) = G(W(x), x). \quad (17)$$

²⁵This algorithm is due to Brooks Ferebee.

and then for every solution \mathbf{a} of (HJB_0) solve the initial value problem

$$\begin{aligned} W'(x) &= G(W(x), x) \\ W(a) &= \int_0^{\infty} e^{-\delta t} f_0(a, u) dt = \frac{1}{\delta} f_0(a, u) \\ 0 &= f(a, u) \end{aligned}$$

3. For every x compute W_i and then use

$$V(x) = \underset{i}{Max} W_i \tag{18}$$

V is the desired value function of our problem.

We want to note that V permits to calculate the optimal control $u(x)$ in feedback form by (HJB) . We also want to remark that at the cut-off point, the Skiba-point, the policy function will not be continuous, see also Wirl and Feichtinger (1999), Santos (1999), and Semmler and Greiner (1999). The main achievement, however, is to find the location of the solutions to (HJB_0) which are optimal steady states.

Our **second method** employs the maximum principle and the associated Hamiltonian. Usually the globally optimal steady states cannot be detected by using the Hamiltonian – since it works with the necessary conditions only. Yet, one can employ the Hamiltonian $H(\cdot)$ associated with the problem $P(a)$ and follow the following three steps.²⁶

1. Compute the candidates for equilibria from control (or co-state) equations and state equations.
2. Compute the local dynamics about the candidates for the steady state where usually there is a unstable candidate for a steady state in the middle and saddle points to the right and to the left of it.
3. Compute the integrals along the stable manifold from the right and from the left. Where the two integral curves intersect represents the cut-off or Skiba point where the domains of attraction separate.

This latter method has been suggested by Skiba(1978) and then further pursued by Brock and Malliaris (1996) and Brock and Starret (1998), but has not been, as to our knowledges, numerically implemented as we do below.

As **third method** dynamic programming can be employed. In our example of growth and credit market the dynamic programming equation

$$H(k, B^*(k)) = \underset{j}{Max} \left[f(k, j) + \left(\frac{d}{dk} B^*(k) \right) G(k, j) \right] \tag{19}$$

differs slightly from the usual one since here iterating on the value function is equivalent to iterating on the maximum amount of debt, B^* , that is sustainable for the agent. The use of dynamic programming on a grid for the state and control equations is, however,

²⁶This method was developed by W. Beyn and T. Pampel from the University of Bielefeld whom we want to thank to make it available to us.

contaminated with numerical rounding errors that pile up in the iteration of the value function. Only if one has reliable estimation on the error bound one can rely on numerical solutions of dynamic programming problem. The discretization problems are discussed in appendix II and an estimation of error bounds is given in Semmler and Sieveking (1998a). Subsequently we also will employ the dynamic programming algorithm but we would like to point out that results obtained from dynamic programming may not be as reliable as the other two methods in finding Skiba-points.

A more rigorous study of the dynamics of a control problem with multiple steady state would require to locate the Skiba-points analytically if there are such. Unfortunately, there do not seem to exist a "local" equation similar to the procedure (HJB_0) to find analytically such cut-off points s . Thus, such cut-off points have to be determined numerically by either of the above three methods.

4 The Growth Model with Credit Market

Next we study a simple growth model which is, similarly to Blanchard (1983) written in efficiency labor, yet instead of maximizing a utility functional, following the transformation as mentioned in section 2, the present value of a net income function is maximized. Computing the present value then is, strictly speaking, only be feasible if there is a constant credit cost factor as in the debt equation (1). In our general case, however, credit cost may be state dependent. Subsequently, we simplify and take $H(k, B) = h(B)$ and study creditworthiness for the univariate case where the capital stock is the only state variable. Yet, multiple equilibria and Skiba points as discussed in section 3 may arise. Employing a growth model as mentioned in section 2 in terms of efficiency labor²⁷ we can write

$$\dot{k} = j - \sigma k, \quad k(0) = k \quad (20)$$

$$\dot{B} = H(k, B) - f(k, j), \quad B(0) = B \quad (21)$$

$$f(k, j) = k^\alpha - j - j^\beta k^{-\gamma}$$

where $\sigma > 0, \alpha > 0, \gamma > 0$ are constants.²⁸ For the simplification $H(k, B) = h(B)$ we assume that h is twice differentiable $h(0) = 0, h' \geq \theta$ for some constant $\theta > 0$ and

²⁷The subsequent growth model can be viewed as a standard RBC model where the stochastic process for technology shocks is shut down and technical change is exogenous occurring at a constant rate. Moreover, a debt equation, as in (21) is added. In Bernanke, Gertler and Gilchrist (1998) net worth is the second state equation. In fact, it can be shown that their use of the second state equation is equivalent to our equ. (21) except for the use of adjustment cost in our model. In our case, however, investment is the (intertemporal) control variable and $H(k, B)$ affects the bifurcation to low and high level equilibria.

²⁸Note that the production function may k^α may have to be multiplied by a scaling factor. For the analytics we leave it aside.

$h'' \geq 0$.²⁹ In the above model $\sigma > 0$ captures both a constant growth rate of productivity as well as a capital depreciation rate and population growth.³⁰ Blanchard (1983) used $h(B) = \theta B, \beta = 2, \gamma = 1$ to analyze optimal indebtedness of a country (see also Blanchard and Fischer 1989, ch. 2).

It is worth noting that the present model has not quite the same mathematical properties as considered in the general case discussed in appendices II and III because first, the range of the control variable j is unbounded, second, the range of the state variables (k, B) are unbounded, and third the net income function $f(k, j)$ has a singularity at $k = 0$. Yet, the present value for the above model can be approximated by one which belongs to the class of models considered in appendices II and III (to do so we can reduce the ranges and smooth f out at $k = 0$).

There are, due to the fact that k is univariate here, simple methods to determine the present value $B^*(k)$ which are much more efficient and explicit than dynamic programming. One algorithm is derived from our first method discussed in section 3 which is based on the observation that the solution $k \rightarrow (k, B^*(k))$ of our problem consists of solutions to a differential equation which (i) use either steepest descent or the least steep ascent in the (k, B) -space and (ii) run into a stationary state.

This simple case where there is a unique common stationary capital k^* for the investment $j_-(k, B)$ with steepest descent as well as for the investment $j_+(k, B)$ with least steep ascent is shown in figure 5. Here we need to decrease the capital k most rapidly while $k > k^*$ using $j_-(k, B)$ and increase capital least possible while $k < k^*$. This we call extremal investment. Of course, in order to ensure that our simulated curve contains $(k^*, B^*(k))$, we invert the time and solve the initial value problem with $(k^*, B^*(k))$ as initial value (using j_- for $k > k^*$ and j_+ for $k < k^*$).

A more complicated situation is shown in figure 6. In addition to the attractor equilibrium k^* there is a non-equilibrium cut off point k^{**} where the solution path $k \rightarrow (k, B^*(k))$ changes direction. As shown in section 3 such cut-off points will turn up in general if several attractor equilibria exist. In the current context the cut-off point, the Skiba point, k^{**} is economically significant as a poverty trap: below k^{**} the country has to reduce its capital to zero in order to keep debt bounded - at least if debt is close to the critical value $B^*(k)$.³¹ Note, however, that such a cut-off point does not have to coincide with a candidate for an equilibrium. This will be demonstrated below. Recall that at a critical debt level $B^*(k)$ the investment (either j_- or j_+) is the optimal investment (the one which realizes the present value of k).

For a constant discount rate we might apply our second method derived from Pontryagin's maximum principle to determine the optimal investment. In the situation of figure 6, we obtain two candidates for an optimal stationary capital, k^*, k^{**} , but would get no

²⁹Note that if $k = 0$ in (21) we have the case of a finance constrained economy (imperfect capital market) as in Brock and Deckert (1985). In our model, however, the adjustment cost is written in a way that we do not need increasing returns as in their paper.

³⁰For details, see Blanchard (1983).

³¹Thus, there exists a development trap as, for example, discussed in Azariadis and Drazen (1990) and Majumdar and Mitra (1995).

information that one of those (and which) is non-optimal.

The general case given by

$$\begin{aligned}\dot{k} &= j - \sigma(k) \\ \dot{B} &= H(k, B) - f(k, j)\end{aligned}$$

is covered by our first algorithm of section 3. This algorithm simply computes $B^*(k)$ as the maximum of all $k \rightarrow (k, B(k))$, considered as functions of k , which satisfy our definition of extremal investment as more precisely defined below.

Let us refer to the simplified version $H(k, B) = h(B)$. Here, since the critical curve $k \rightarrow B^*(k)$ is tangent to $B^*(k)$, we can consider the set S of states in the (k, B) -space where investments $j = \sigma k$ decrease debt.

Investment keeps capital constant precisely if

$$h(B) \leq f(k, \sigma k) \quad \text{or}$$

$$B \leq h^{-1}(f(k, \sigma k)) =: \varphi(k).$$

Let

$$S = \{(k, B) \mid 0 \leq B \leq \varphi(k), \quad k \geq 0\}$$

S is bounded by the graph of the function φ , which we define only for $k \geq 0$ with $f(k, \sigma k) \geq 0$.

The following figure shows different possible shapes of S depending on the value of parameter γ .

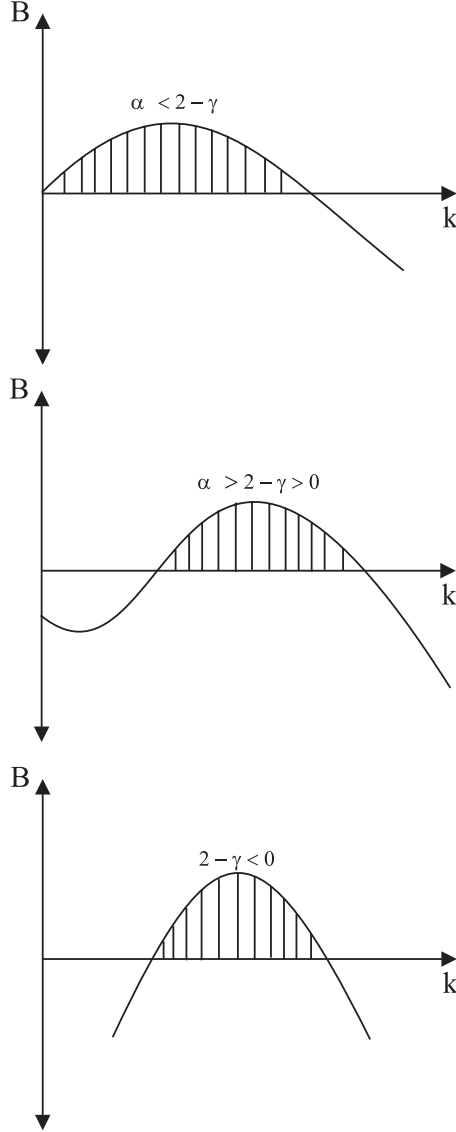


Figure 2: possible shapes of the region S of (k, B) where $h(B) - f(k, \sigma k) \leq 0$.

For the general case we have the following

Definition of Extremal Investment Let $H, f : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma : [0, +\infty)$ be continuous functions, $f(k, 0) = 0$ for all $k \geq 0$, $f(k, s)$ continuously differentiable with respect to j and $\frac{\partial}{\partial j} f(k, j) < 0$. Suppose $H(k, B) > f(k, \sigma(k))$. There exists $j_+ = j_+(k, B)$ and $j_- = j_-(k, B)$ such that

$$W_-(k, B) := \frac{H(k, B) - f(k, j_-)}{j_- - \sigma(k)} = \text{Max} \left\{ \frac{H(k, B) - f(k, j)}{j - \sigma(k)} \mid j - \sigma(k) < 0 \right\}$$

$$W_+(k, B) := \frac{H(k, B) - f(k, j_+)}{j_+ - \sigma(k)} = \text{Min} \left\{ \frac{H(k, B) - f(k, j)}{j - \sigma(k)} \mid j - \sigma(k) > 0 \right\}$$

j_- and j_+ are called extremal investments. The corresponding extremal vector fields are defined by $v_{\pm}(k, B) = (j_{\pm}(k, B) - \sigma(k), H(k, B) - f(k, j_{\pm}(k, B)))$. For $H(k, B) = f(k, \sigma(k))$ put $j_{\pm}(k, B) = \sigma(k)$. Let j_- and j_+ respectively be the investment which produces the steepest descent or the least steep ascent in the (k, B) -space, $j = j_{\pm}$, and satisfies

$$\frac{H(k, B) - f(k, j)}{j - \sigma(k)} = -\frac{\partial}{\partial j} f(k, j).$$

Note that the cut-off point where domains of attraction separate will depend on the shapes of the credit cost function $H(k, B)$ and the net income function $f(k, j)$. The solution to the last equation for our simplified functions $H(k, B) = h(B)$, $f(k, j) = k^{\alpha} - j - j^2 k^{-\gamma}$, $\sigma(k) = \sigma k$ is that of a quadratic equation:

$$j_+(k, B) := \sigma k + \sqrt{\sigma^2 k^2 + \sigma k^{1+\gamma} + h(B)k^{\gamma} - k^{\alpha+\gamma}} \quad (22)$$

with

$$v_+(k, B) := \left(1, \frac{h(B) - f(k, j_+)}{j_+ - \sigma k}\right) = (1, 1 + 2j_+ k^{-\gamma}).$$

The investment rate giving the least slope to

$$-\left(1, \frac{h(B) - f(k, j)}{j - \sigma k}\right)$$

while decreasing capital is (similarly)

$$j_-(k, B) := \sigma k - \sqrt{\sigma^2 k^2 + \sigma k^{1+\gamma} + h(B)k^{\gamma} - k^{\alpha+\gamma}} \quad (23)$$

with

$$v_-(k, B) := \left(1, \frac{h(B) - f(k, j_-)}{j_- - \sigma k}\right) = (1, 1 + 2j_- k^{-\gamma}).$$

Note that if $h(B) = f(k, \sigma k)$ that is at the boundary of S :

$$v_+(k, B) = v_-(k, B) = (1, 1 + 2\sigma k^{1-\gamma})$$

Note also that $v_+(k, B)$ and $v_-(k, B)$ are tangent to S .

Proposition 1 Equilibria satisfy $1 + 2\sigma k^{1-\gamma} = \varphi'(k)$.

For $h(B) = rB^{\kappa}$ this is equivalent to

$$1 + 2\sigma k^{1-\gamma} = \frac{\alpha k^{\alpha-1} - \sigma - \sigma^2(2-\gamma)k^{1-\gamma}}{r^{1/\kappa} \kappa (k^{\alpha} - \sigma k - \sigma^2 k^{2-\gamma})^{(\kappa-1)/\kappa}} \quad (24)$$

For a constant credit cost factor one can, using our second method, obtain the equilibria also through the maximum principle and the Hamiltonian.

It is useful to look at the B -isocline of v_- that is the set of $k > 0, B > 0$ where $1 + 2j_-(k, B)k^{-\gamma} = 0$. Simple calculation shows that the latter equality is equivalent to

$$B = h^{-1}\left(k^\alpha + \frac{1}{4}k^\gamma\right)$$

The following figure shows a sketch of the phase portrait of v_- based on this equation.

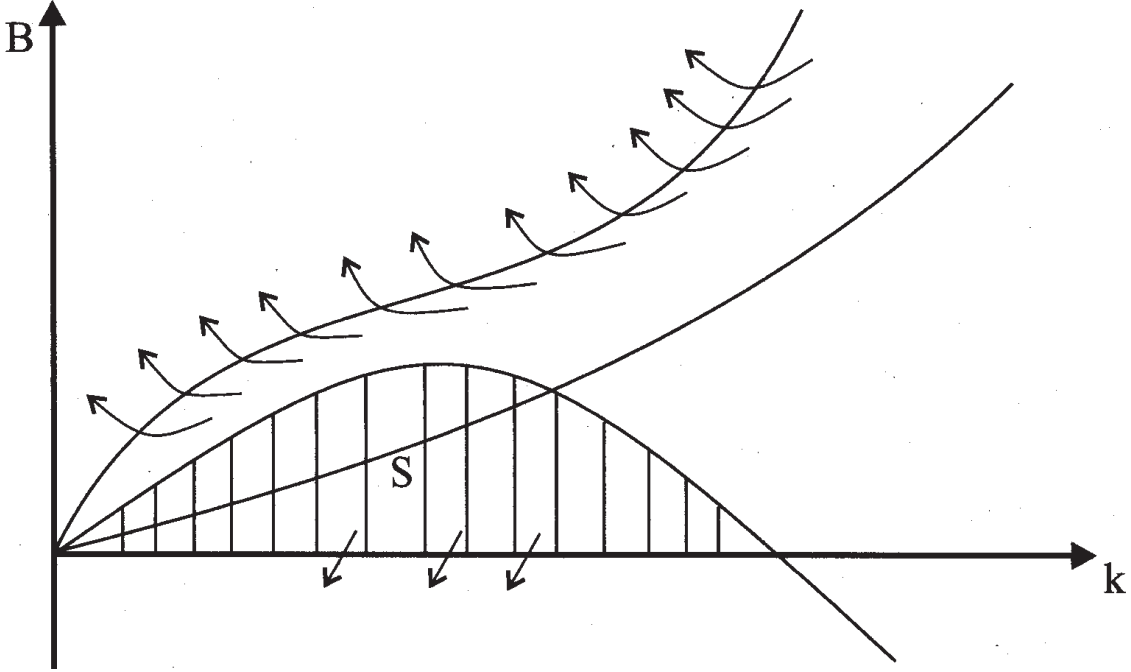


Figure 3: sketch of the phase portrait of v_- near the origin based on the set where $\dot{B} = 0$.

The B -isocline of v_- lies completely above S . Through every point of it there is a unique trajectory for v_- . These trajectories may not be defined for all $k \geq 0$; this happens if they run into S .

Consider, however,

Case 1 All v_- trajectories crossing the B isocline are defined for all $k \geq 0$.

In this case these trajectories admit on infimum $k \rightarrow (k, B_1(k))$ which satisfies

$$\lim_{k \rightarrow \infty} B_1(k) = B_1(0) = 0$$

B_1 , in Case 1, is defined for all $k \geq 0$ and solves our debt control problem:

Proposition 2 Suppose all trajectories through the B -isocline of v_- are defined for all $k \geq 0$ (i.e. they lie above S). Then their infimum $k \rightarrow (k, B_1(k))$ solves the debt control problem.

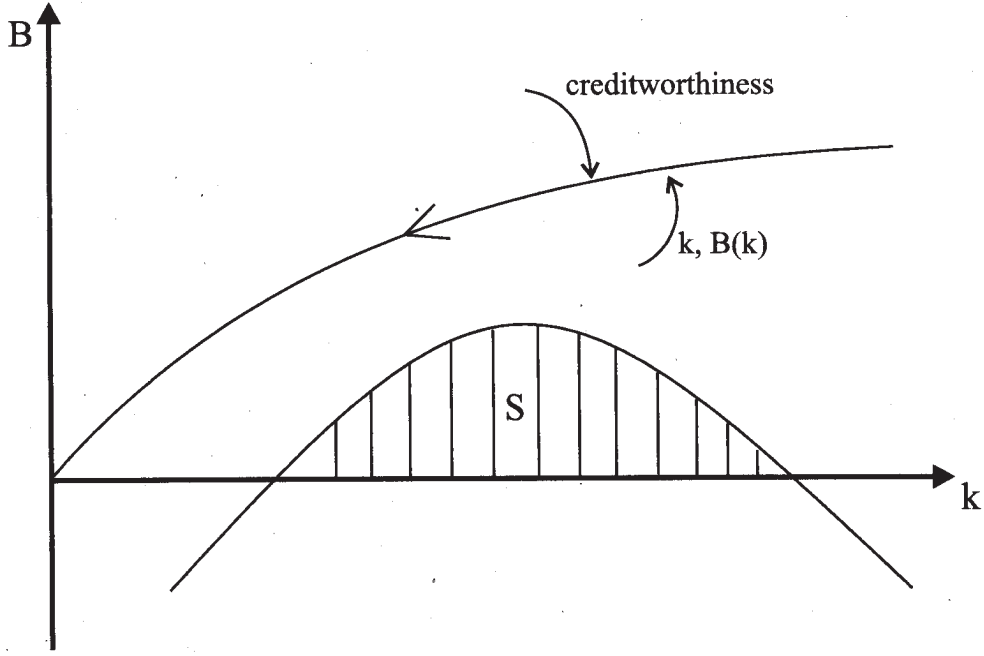


Figure 4: Creditworthiness in Case 1.

Proof of Proposition 2 It is obvious that B may be held bounded if it is initially below $B_1(k)$. Therefore $B_1(k) \leq B^*(k)$. Now Suppose $B(0) < B^*(k)$, the argument of Lemma 2 of appendix II shows that there is an investment function $j(t)$ which steers $B(t) < 0$ for some finite $t > 0$. This, however, is impossible for $B(0) > B_1(k)$ since no solution of our differential equations may cross $k \rightarrow (k, B_1(k))$ from above. Therefore, $B(0) < B_1(k)$. Since $B(0) < B^*(k)$ was arbitrary $B^*(k) \leq B_1(k)$ q.e.d.

Case 2a One of the v_- trajectories which cross the B isocline of v_- runs into S . Consider the infimum $k \rightarrow (k, B_2(k))$ of all v_- trajectories defined for all $k \geq 0$, i.e. above S . It is tangent at S in at most on point. Let (k^*, B^*) be the one with largest first component among all points in which $(k, B_2(k))$ is tangent at S .

There is a v_+ trajectory $k \rightarrow (k_1 B_3(k))$ through (k^*, B^*) which is also tangent at S in (k^*, B^*) . v_+ trajectories may cross v_- trajectories only from below (by construction). Therefore

$$B_3(k) \leq B_2(k) \quad \text{for } 0 \leq k \leq k^*.$$

Set

$$B_4(k) = \begin{cases} B_3(k) & \text{for } 0 \leq k \leq k^* \\ B_2(k) & \text{for } k^* \leq k \end{cases}$$

Assume that $B_3(0) = \lim_{k \rightarrow 0} B_3(k) \geq 0$.

Proposition 3 $B^*(k) = B_4(k)$ in case 2a where $B_3(k) \geq 0$.

Proof Obvious from the following figure, like the proof of Proposition 2.

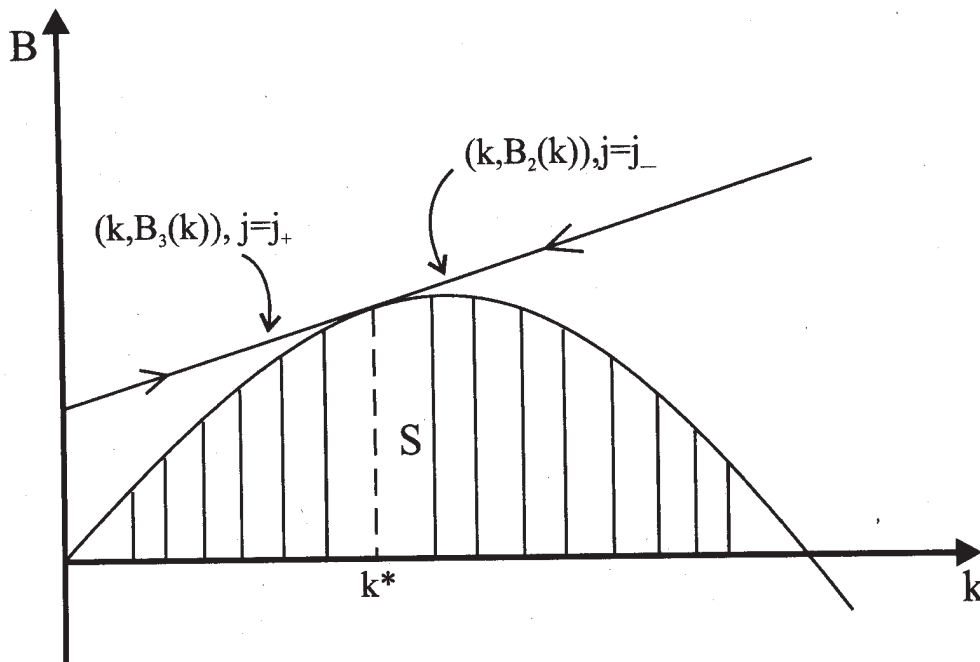


Figure 5: The dotted line is creditworthiness; above k^* debt is decreased with j_- , below k^* debt is increased with j_+ .

Case 2b Assume $B_3(k) = 0$ for some positive k , $0 \leq k < k^*$. In this case there is a v_- trajectory $k \rightarrow (k, B_5(k))$ such that $B_5(0) = 0$ and $B_5(k^{**}) = B_3(k^{**})$ for uniquely determined k^{**} between zero and k^* , k^{**} is a cut-off point, but not an equilibrium for debt control: below k^{**} capital and debt are decreased, above k^{**} capital and debt are increased up to k^* . Set

$$B_6(k) = \begin{cases} B_5(k) & \text{for } 0 \leq k \leq k^{**} \\ B_3(k) & \text{for } 0 \leq k^{**} \leq k^* \\ B_2(k) & \text{for } k^* \leq k \end{cases}$$

Proposition 3 $B^*(k) = B_6(k)$ in case 2a.

The proof of Proposition 3, following that of Proposition 1 is obvious from the following figure:

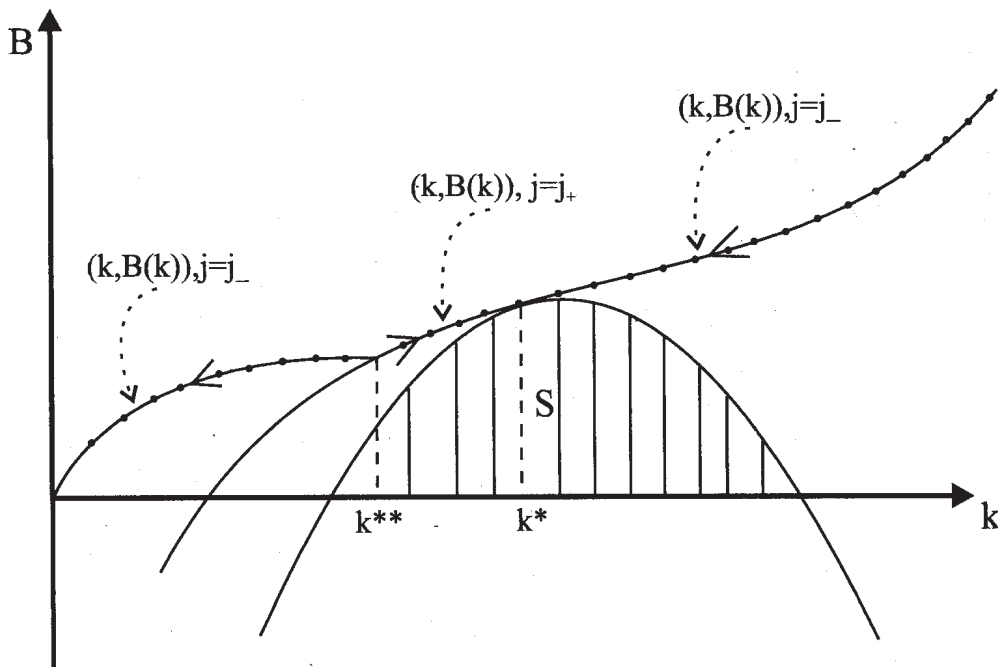


Figure 6: The dotted line is the creditworthiness curve. Investment strategies along this curve change according to j_-, j_+, j_- .

Note that in order to reduce debt in Case 2 it may be necessary to first increase it. This happens if the capital stock level $k < k^*$ is too small, i.e. the economy not developed sufficiently. Moreover, there is a threshold, k^{**} , below which the country cannot take off even if the country can freely borrow.

5 The Numerical Study of the Model

Next we apply the first type of algorithm to compute the critical debt. Its inputs are three functions H, f, σ as in the definition of the extremal investments. Its output is the critical debt $B^*(k)$, i.e. the negative of the present value of k . The algorithm works if we assume that the vector fields $v_{\pm}(k, B)$ admit but a finite number of singularities $v_{\pm}(k, B) = 0$ and that for any initial capital k the optimal investment (i.e. the one which realizes its present value) steers the capital stock towards an equilibrium. Our first algorithm which admits nonlinear credit cost computes the critical curve below which the debt can be steered bounded.

We compute the critical curve for the above example of a univariate capital stock with nonlinear adjustment cost for investment and state dependent credit cost. Due to the nonlinear adjustment cost the model may exhibit multiple candidates for equilibria as discussed above and as depicted in Figure 6. A case like this will be numerically studied below.

We will compare our results with the results obtained from the Hamiltonian equation

and our second method and the dynamic programming algorithm proposed in section 3.

A more general form of the first algorithm can be stated for our non-standard case where the credit cost is state dependent. The critical value problem can be computed as follows.

1. Compute all solutions (k_i^+, B_i^+) $i = 1, 2, \dots$ and (k_j^-, B_j^-) $j = 1, 2, \dots$ to $v^+(k, B) = 0$ and $v^-(k, B) = 0$ respectively.
2. Compute all solutions $B_i^+(\cdot)$ and $B_i^-(\cdot)$ to the critical value problems

$$\frac{d}{dk}B(k) = -w^+(k, B), \quad k \leq k_i^+, \quad B(k_i^+) = B_i^+ \quad i = 1, 2, \dots$$

respectively to

$$\frac{d}{dk}B(k) = -w^-(k, B), \quad k \geq k_i^-, \quad B(k_i^-) = B_i^- \quad i = 1, 2, \dots$$

3. Compute $B^*(k)$ as the maximum of all $B(k_i^+)$ and $B(k_i^-)$ respectively.

In our numerical example we employ the function

$$f(k, j) = k^\alpha - j - j^2 k^{-\gamma} - c$$

and

$$h(B) = rB^\kappa$$

We compute the candidates for equilibria for our debt control problem (20) - (21) both, first, by employing the Hamiltonian for $\kappa = 1$ and, second, by using the tangency condition (24) from our vector field analysis with $\kappa \geq 1$.

Following our second method, we denote x as the co-state variable in the Hamiltonian equations in appendix I. The function $f(k, \cdot)$ is strictly concave by assumption therefore there is a function $j(k, x)$ which satisfies the first order condition of the Hamiltonian

$$f_j(k, j) + x = 0$$

and $j(\cdot, \cdot)$ is uniquely determined thereby. It follows that (k, x) satisfy

$$\dot{k} = j(k, x) - \sigma k \tag{25}$$

$$\dot{x} = (\sigma + \theta)x - f_k(k, j(k, x)) \tag{26}$$

The isoclines can be obtained by the points in the (k, x) space where $\dot{k} = 0$ satisfies

$$x = 1 + 2\sigma k^{1-\gamma} \tag{27}$$

and where $\dot{x} = 0$ satisfies

$$x_{\pm} = 1 + \vartheta k^{1-\gamma} \pm \sqrt{\vartheta^2 k^{2-2\gamma} + 2\vartheta k^{1-\gamma} - 4\alpha\gamma^{-1} k^{\alpha-\gamma}} \tag{28}$$

where $\vartheta = 2\gamma^{-1}(\sigma + \theta)$. Note that the latter isocline has two branches.

In our numerical example we compute the equilibrium candidates through the Hamiltonian equations (25) - (26) and through the tangency condition (24). We scale the production function by a^{32} and take $c = 0$. We employ the following parameters: $\alpha = 1.1$, $\gamma = 0.3$, $\sigma = 0.15$, $\theta = 0.1$, $r = 0.1$. Moreover, we take $\kappa = 1$ for the Hamiltonian and $\kappa = 1.05$ for equation (24).

For those parameters, using the Hamiltonian approach, Figure 7 depicts the isoclines (27) - (28) showing two positive candidates for equilibria.

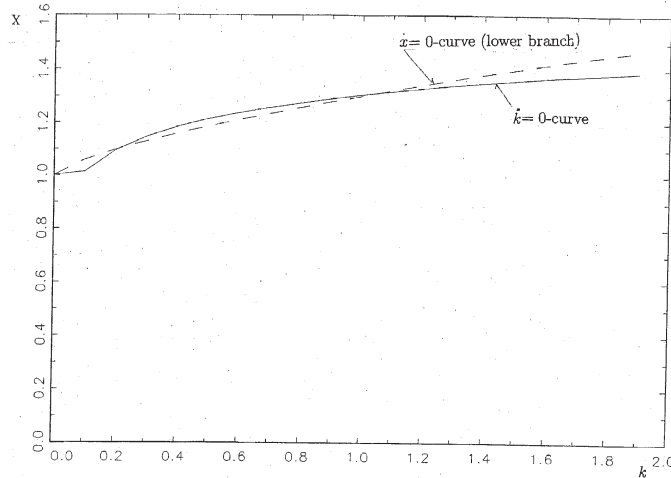


Figure 7: Isoclines and equilibria from the Hamiltonian equation

Figure 7:

The two equilibrium candidates are: (HE1): $k^* = 1.057$, $x^* = 1.3$ and (HE2): $k^{**} = 0.21$, $x^{**} = 1.1$. The two candidates are numerically obtained by using a nonlinear equation solver.³³ Since the second branch of (28) does not intersect with (27) we have left it aside. We also want to note that the equilibrium candidate (k^*, x^*) is a saddle whereas (k^{**}, x^{**}) represents a repeller. We want to stress again, as in our first method, that from the Hamiltonian equation one only obtains candidates for equilibria.

The candidates for equilibria can also be computed for $\kappa \geq 1$ using the tangency condition (24) from the first method. Figures 8a and 8b show the two candidates, for $\kappa = 1.05$ in Figure 8a, and for $\kappa = 1.25$ in Figure 8b.

For $\kappa = 1.05$ two equilibria are obtained through (24) by using the nonlinear equation solver. The candidates are (VE1): $k^* = 0.69$, $B^* = 0.79$, and (VE2): $k^{**} = 0.13$, $B^{**} = 0.14$. Employing, however, $\kappa = 1.25$ only one candidate remains. The left one has disappeared and the candidate to the right remains.³⁴

³²We have multiplied the production function by $a = 0.29$ in order to obtain sufficiently separated equilibria.

³³The nonlinear equation solver from the software package GAUSS is employed.

³⁴We also want to note that for $\kappa = 1$ both methods to compute the equilibria give the same numerical results.

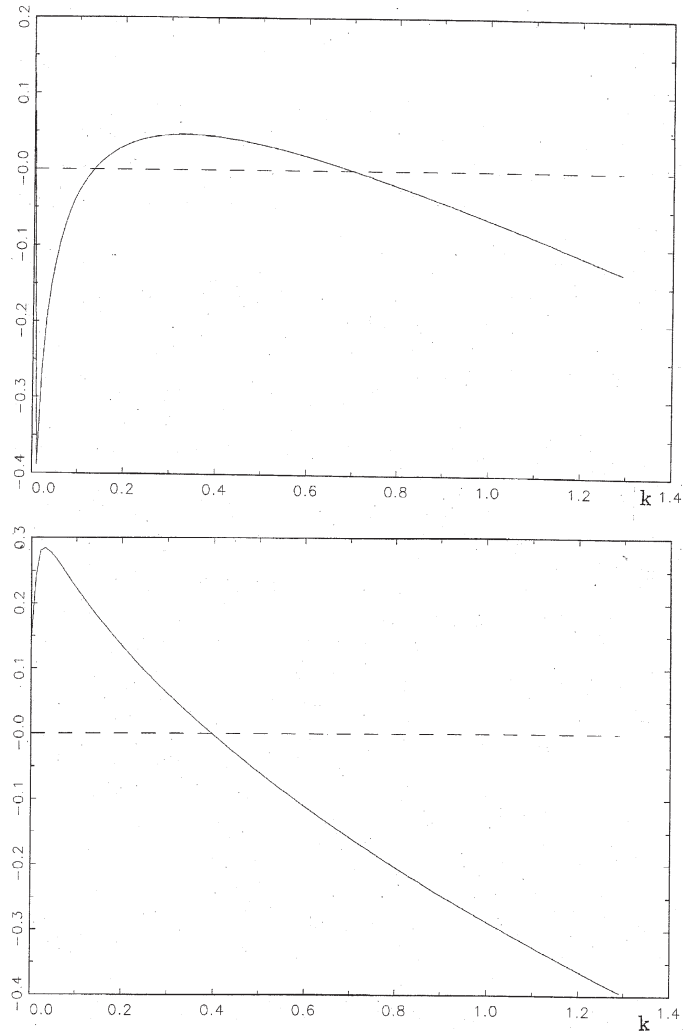


Figure 8: Equilibria computed through (24) for $\kappa = 1.05$ (above) and $\kappa = 1.25$ (below).

Next we use the above stated first type of algorithm and compute the solutions to the critical value problem. The curve $h(B) = f(k, j)$ and the critical curve for the case $\kappa = 1.05$, starting at the candidate $k^* = 0.69$, $B^* = 0.79$, are depicted in Figure 9.

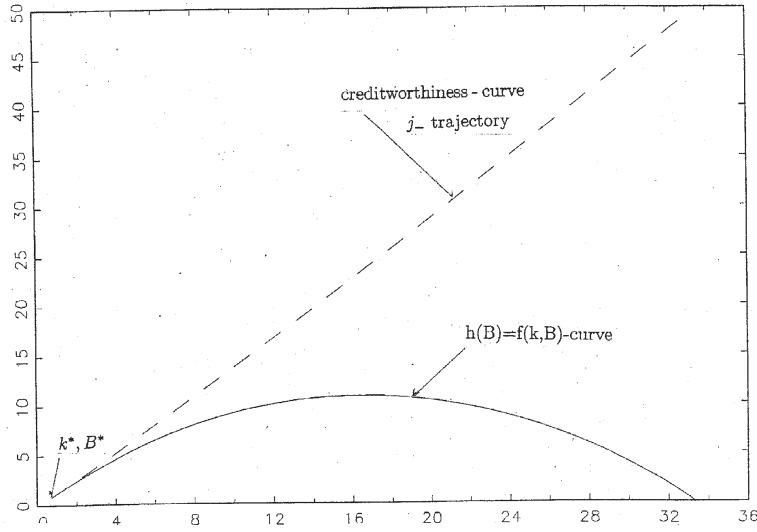


Figure 9: The $h(B) = f(k, j)$ curve and the creditworthiness-curve (the j_- trajectory running into equilibrium $k^* = 0.69$, $B^* = 0.79$).

Applying again the above suggested algorithm starting from $k^* = 0.69$, $B^* = 0.79$, pursuing the j_+ trajectory, and starting from the origin $k = 0, B = 0$, pursuing the j_- trajectory we obtain the graphs as depicted in Figure 10.

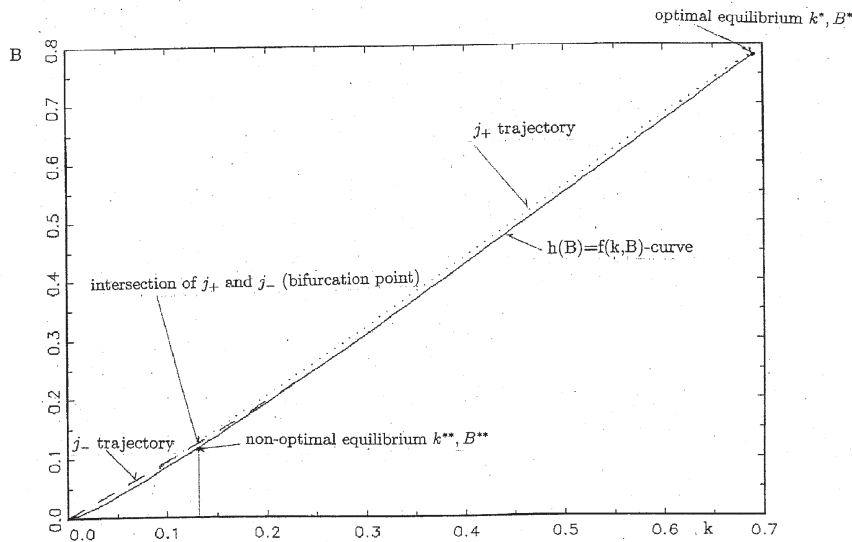


Figure 10: The creditworthiness-curve as the maximum of the j_+ trajectory (starting from $k^* = 0.69$, $B^* = 0.79$) and the j_- trajectory (starting from $k = 0, B = 0$).

Note that there is a cut-off point where the two trajectories intersect. There the optimal investment either leads to $k^* = 0.69$, $B^* = 0.79$ or to $k = 0, B = 0$. Note

also that the equilibrium candidate $k^{**} = 0.13$, $B^{**} = 0.14$ is non-optimal, since there are investment strategies that improve the countries wealth. On the other hand the equilibrium candidate $k^* = 0.69$, $B^* = 0.79$ is optimal.

We have also computed the critical curve through the use of our third method, the dynamic programming algorithm as proposed by Sieveking and Semmler (1997). This is depicted in Figure 11.

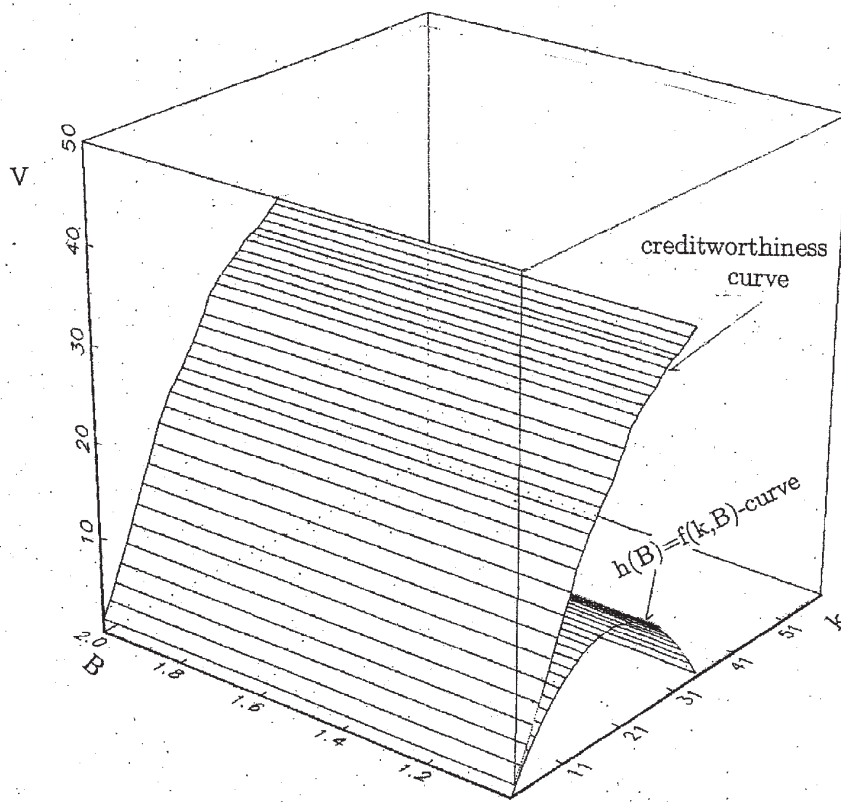


Figure 11: The creditworthiness - curve computed through a dynamic programming algorithm.

Although the dynamic programming algorithm is capable of computing the creditworthiness - curve the above suggested first two algorithms are substantially more efficient to do so. For the two algorithm no iteration on the value function through the choice of a grid size, as required by the dynamic programming algorithm, is necessary. Errors from the discretization, arising from grid size, rounding errors and keeping a piecewise constant control – which may affect the dynamic programming solution – do not occur in our above first two methods. It will be difficult to detect a cut-off point such as shown in Figure 10 by the use of a dynamic programming algorithm since the cut-off point for a change of control may be affected by the numerical errors.

Finally, we have computed the critical curve for $\kappa = 1$ by using the Hamiltonian

equations. Here, we use an algorithm that is able to compute the stable manifold of the saddle point $k^* = 1.057$, $x^* = 1.3$ and the integral along the stable manifold.³⁵ The dynamics about the two candidates of the equilibria are depicted in Figure 12

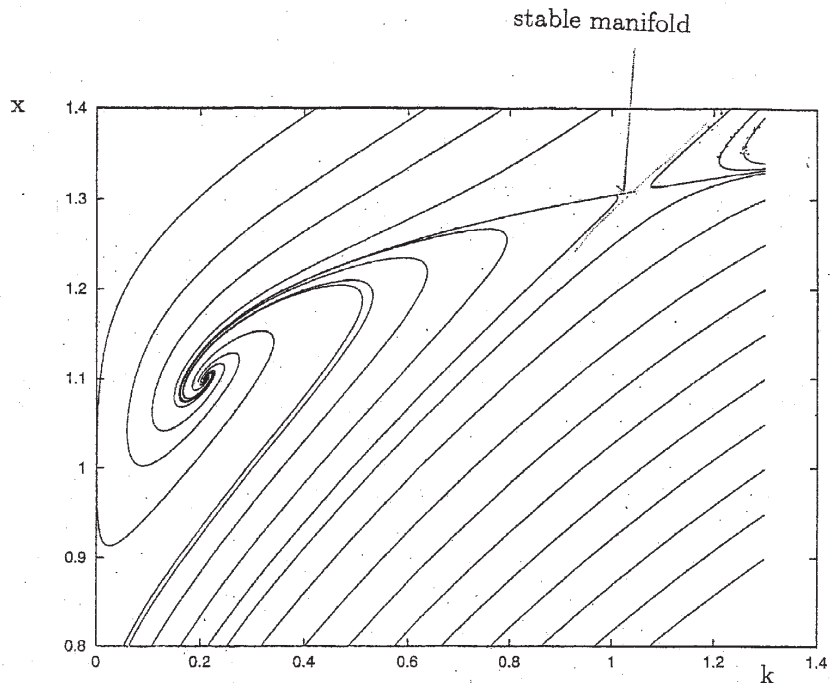


Figure 12: Stable manifold passing through the saddle point $k^* = 1.057$, $x^* = 1.3$.

with the dotted line, starting at $k^{**} = 0.21$, $x^{**} = 1.1$ and moving into the equilibrium candidate $k^* = 1.057$, $x^* = 1.3$, representing the stable manifold.

³⁵This procedure that follows the literature since Skiba (1978) was developed by Wolf-Jürgen Beyn and Thorston Pampel from the Dept. of Mathematics, University of Bielefeld. We want to thank them for making it available to us.

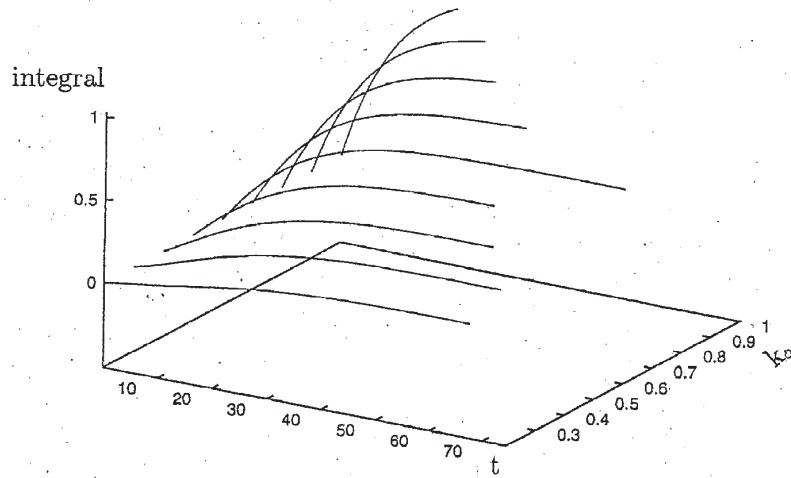


Figure 13: The creditworthiness - curve computed through the integral along the stable manifold.

Figure 13 shows the value of the integral starting with the initial condition $k_0 = 0.21$ going up to $k_0 = 1.0$. The value of the integral represents points on the creditworthiness - curve for each initial condition, k_0 , to the right of the equilibrium candidate k^{**} .

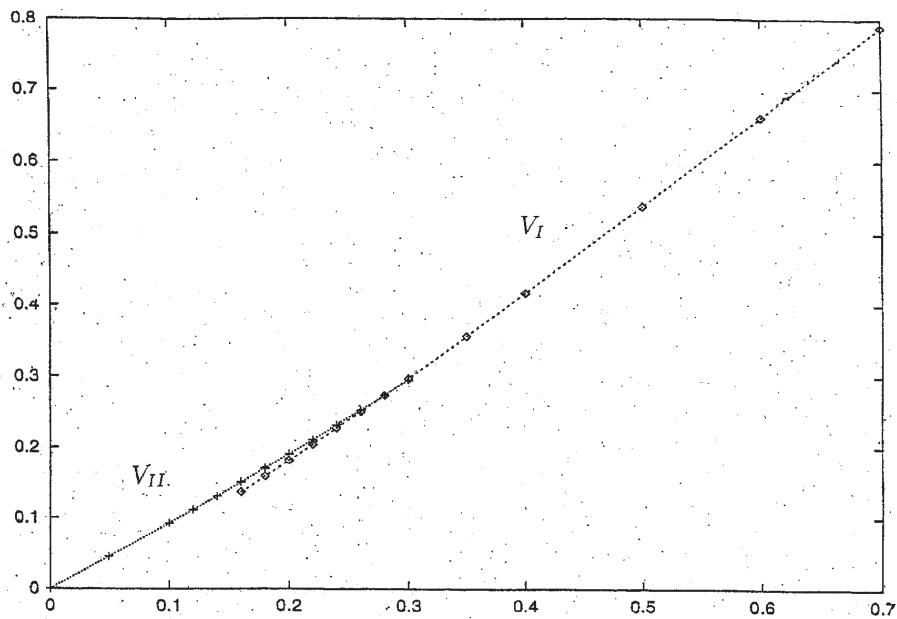


Figure 14: The creditworthiness - curve and the cut-off point for the two solutions.

Figure 14 represents the two integral lines and the cut-off point. The two integral lines are obtained by starting with k_0 and running either into equilibrium candidate $k^* = 1.057$, V_I (stable manifold) or running from the unstable candidate k^{**} into $k = 0$. The latter is

V_{II} . For $\kappa = 1$, and thus for particular fixed interest rate, the Hamiltonian equations can also be employed to compute the critical curve. For more general debt pay off functions, however, our first method appears to be suited better. Yet, both methods are able to demonstrate that cut-off points do not necessarily coincide with candidates for equilibria.

6 Estimating the Parameters of the Model

Next, we want to take our growth model with adjustment cost of investment to the data. We will use quarterly data from Euroland encompassing its core countries. For the purpose of the parameter estimation we have to transform our dynamic equations into estimable equations. By presuming the univariate version where in the debt equation only a constant credit cost factor enters we can employ the Hamiltonian equation. This is in the case of Euroland justified, since there are likely to be not severe idiosyncratic risk components in the interest rate. We can transform the system (A3)-(A4) of appendix I into estimable equations and employ time series data on capital stock and investment – all expressed in efficiency units – to estimate the involved parameter set.³⁶

Substituting from the appendix I (A2) into (A3) we get the following two dynamic equations

$$\dot{k} = \left(\frac{x-1}{k^{-\gamma} \cdot \beta} \right)^{\frac{1}{1-\beta}} - \sigma k \quad (29)$$

$$\dot{x} = (\sigma + \theta)x - \alpha k^{\alpha-1} - j^{\beta} \gamma k^{(-\gamma-1)} \quad (30)$$

Next, we transform the above system (29)-(30) into observable variables so that we obtain estimable dynamic equations.

From (29) we obtain

$$\hat{k} = j/k - \sigma \quad (31)$$

with $\hat{k} = \dot{k}/k$

Note that from (A1) in appendix I we can get

$$x = 1 + \beta j^{\beta-1} k^{-\gamma} \quad (32)$$

Taking the time derivative of (32) we obtain

$$\dot{x} = (\beta(\beta-1)j^{\beta-2}k^{-\gamma}) \cdot \dot{j} \quad (33)$$

and using (30) we have

³⁶Estimable equations for a version with a state dependent debt service as in equ. (21) would predict a slightly different paths for optimal investment and capital stock namely such as given by (22) and (23) where also the nonlinear credit cost $H(k, B)$ enters. Those equations, however, appear to be more cumbersome to estimate.

$$(\beta(\beta - 1)j^{\beta-2}k^{-\gamma}) \cdot \dot{j} = (\sigma + \theta)x - \alpha k^{\alpha-1} - j^{\beta}\gamma k^{(-\gamma-1)}$$

Thus

$$\dot{j} = \frac{(\sigma + \theta)x - \alpha k^{\alpha-1} - j^{\beta}\gamma k^{(-\gamma-1)}}{\beta(\beta - 1)j^{\beta-2}k^{-\gamma}} \quad (34)$$

or

$$\hat{\frac{\dot{j}}{j}} = \left(\frac{(\sigma + \theta)x - \alpha k^{\alpha-1} - j^{\beta}\gamma k^{(-\gamma-1)}}{\beta(\beta - 1)j^{\beta-2}k^{-\gamma}} \right) / j \quad (35)$$

Substituting (32) into (35) we get as estimable equations in observable variables (31) and (35) which depend on the following parameter set to be estimated.

$$\varphi = (\theta, \sigma, \beta, \gamma, \alpha, a)$$

The estimation of the above parameter set is undertaken by aggregating capital stock and investment for the core countries of Euroland. The data are quarterly data from 1978.1 - 1996.2. Although aggregate capital stock data starting from 1970.1 are available, we apply our estimation to the period 1978.1 - 1996.2, since the European Monetary System has been introduced in 1978 whereby the exchange rates between the countries were fixed within a band. This makes the across country aggregation of capital stock and investment feasible. The aggregate capital stock series is for gross private capital stock and the investment series is total fixed investment. Both are taken from OECD data base (1999). The series for gross capital stock and investment represent aggregate real data for German, France, Italy, Spain, Austria, Netherland and Belgium. Since we are employing a model in efficiency labor each countries time series for capital stock and investment is scaled down by labor in efficiency units measured by the time series $L_t = L_0 e^{(n+g_y/l)t}$ where n is average population growth and g_y/l average productivity growth. As to our estimation strategy we employ NLLS estimation and use a constrained optimization procedure.³⁷ The results are shown in Table 1.

Table 1: Parameter estimates for Euroland (1978.1-1996.2)

θ	σ	β	γ	α	a
0.035	0.092	0.312	0.116	0.385	3.32

The parameters obtained from historical data are quite reasonable.³⁸ Overall one can observe that the adjustment cost of investment are not very large since the exponents β and γ are small.

³⁷The estimations were undertaken in GAUSS for which the constrained optimization procedure recently provided by GAUSS was used.

³⁸We want to note that standard errors could not be recovered since the Hessian in the estimation was not non-negative definite.

Using the estimated parameters one can again compute through (25) - (26) the steady states for the capital stock. Doing so numerically it turns out that for our parameter estimates of Table 1 the steady state is unique and we obtain a $k^* = 37.12$ which coincides roughly with the mean of the historical series of the capital stock for Euroland. This gives a steady state of net income of $f(k, j) = 8.799$, computed from (A2) of appendix 1 at the steady state of $k^* = 37.12$. Moreover, for the present value of the net income at the steady state we obtain $V(k^*) = 244.4193$.

Using the estimated parameters figure 15 shows the computed output, investment (including adjustment cost of investment) and the net income.

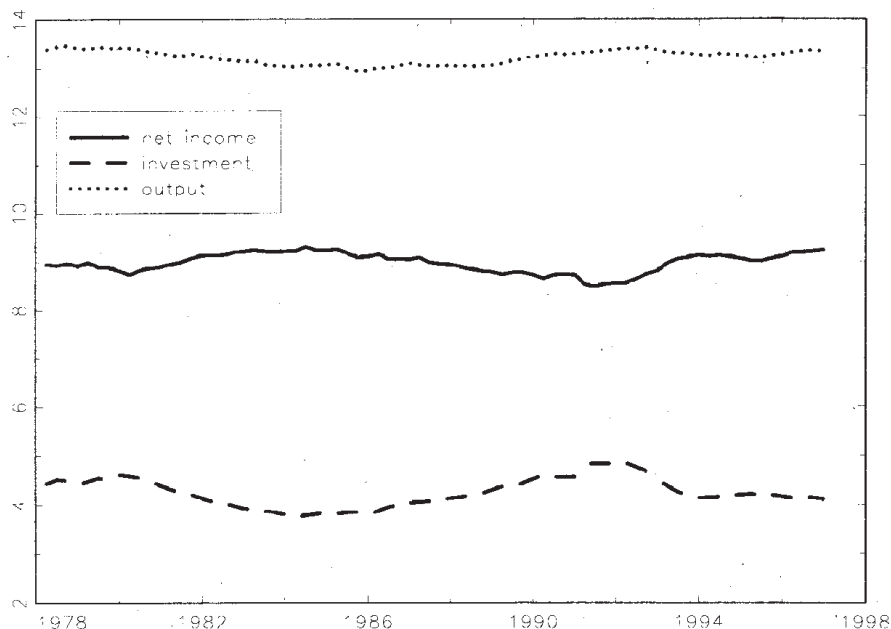


Figure 15: Net income, investment (incl. adjustment cost) and output.

As the figure 15 shows, since we are using the aggregate variables in efficiency units, the output in efficiency unit tends to be stationary and the net income moves inversely to investment (the latter including adjustment cost).

Finally, note that with those parameter estimates given in Table 1 we also can now easily compute the present value outside the steady state and thus the critical debt curve by using either of our above three methods– the HJB equation, the Hamiltonian equation or dynamic programming (whereas, as noted above, the first two methods appear to easier to use). Since, however, there is no external debt of Euroland but rather external assets, as shown in the next section, the result of such an exercise will not be very instructive. The balance sheets of banks and firms, as discussed in Krugman (1999) and Mishkin (1998), will presumably show no sign of deterioration, since Euroland has net claims vis-a-vis the rest of the world. Our methods to compute present value of net income could, however, be fruitfully undertaken for other countries with external debt and balance sheets of banks

and firms deteriorating.³⁹ Note, however, that the above method gives us only asymptotic results, i.e. if $t \Rightarrow \infty$. Next, for Euroland we pursue another method – for finite number of observations – to compute the sustainability of external assets.

7 Testing Sustainability of Debt

Next, following Flood and Garber (1980), Hansen and Sargent (1981) and Hamilton and Flaven (1986) a NLLS estimate for the sustainability of external debt can be designed for a finite number of observations. Similarly to the computation of the capital stock and investment for our core countries of Euroland we have computed the trade account, the current account and the net foreign assets of those core countries for the time period 1978.1-1998.1. Since we want to undertake sustainability tests for certain growth regimes, we have computed monthly observations. In our computation we had to eliminate the trade among the Euroland-countries.⁴⁰ We consider the time series for the entire period 1978.1-1998.12 and in addition subdivide the period into two periods 1978.1-1993.12. and 1994.1-1998.12. The break in 1994 makes sense since the exchange rate crisis of September 1992 lead to a reestablishment of new exchange rates with a wider band in 1993. Thus, the sustainability tests will be undertaken for those two subperiods.

In a discrete version the foreign debt can be computed as follows. Starting with initial debt B_0 one can compute in a discrete time way the stock of debt as follows. By assuming a constant interest rate we have

$$B_t = (1 + r_{t-1})B_{t-1} - TA_t \quad (36)$$

where TA_t is the trade account and B_{t-1} the stock of foreign debt at period $t - 1$ and r_{t-1} the interest rate. As interest rate we took the Libor rate. The initial stock of foreign debt B_0 for 1978.1 has been estimated. This way, the entire time series of external debt and trade account could be generated.

From equ. (36) we can develop a discrete time sustainability test. For reason of simplicity let us assume a constant interest rate. Equ. (36) is then a simple first order difference equation that can be solved by recursive substitution forward leading to

$$B_t = \sum_{i=t+1}^N \frac{TA_i}{(1+r)^{i-t}} + \frac{(1+r)^t B_N}{(1+r)^N} \quad (37)$$

³⁹Of course, one would have to consider also the exchange rate regime under which the country borrows and in particular the fact whether the country (banks, firms) borrows in foreign currency. In this case a exchange rate shock will exacerbate the deterioration of the balance sheets, see Mishkin (1998) and Krugman (1999).

⁴⁰A similar attempt to compute external debt of countries and regions, following a similar methodology as suggested above, has been recently undertaken by Lane and Melesi-Ferreti (1999). Their results for the Euroland core countries show similar trends as our computation. Their results are, however, less precise since they do not eliminate intra-Euroland trade.

In the equ. (37) the second term must go to zero if the intertemporal budget constraint is supposed hold. Then equ. (37) means that the current value of debt is equal to the expected discounted future trade account surplus

$$B_t = E_t \sum_{i=t+1}^{\infty} \frac{TA_i}{(1+r)^{i-t}} \quad (38)$$

Equivalent to requiring that equ. (38) must be fulfilled is that the following condition holds

$$E_t \lim_{N \rightarrow \infty} \frac{B_N}{(1+r)^N} = 0 \quad (39)$$

The equation is the usual transversality condition or no-ponzi game condition as discussed in section 2..

If the foreign debt is constrained not to exceed a constant, A_0 , on the right hand side of (37), we then have

$$B_t = E_t \sum_{i=t+1}^{\infty} \frac{TA_i}{(1+r)^{i-t}} + A_0(1+r)^t \quad (40)$$

The NLLS test proposed by Flood and Garber (1980) and Hamilton and Flaven (1986) and Greiner and Semmler (1998) can be modified for our case. It reads:

$$TA_t = b_1 + b_2 TA_{t-1} + b_3 TA_{t-2} + b_4 TA_{t-3} + \varepsilon_{2t} \quad (41)$$

$$\begin{aligned} B_t &= b_5 (1+r)^t + b_6 + \frac{(b_2 b + b_3 b^2 + b_4 b^3) TA_t}{(1 - b_2 b - b_3 b^2 - b_4 b^3)} \\ &+ \frac{(b_3 b + b_4 b^2) TA_{t-1}}{(1 - b_2 b - b_3 b^2 - b_4 b^3)} + \frac{(b_4 b) TA_{t-2}}{(1 - b_2 b - b_3 b^2 - b_4 b^3)} + \varepsilon_{1t} \end{aligned} \quad (42)$$

We want to note, however, that following Wilcox (1989) it might be reasonable to compute trade account surplus and debt series as discounted time series. We have also undertaken the computation of the those discounted time series by discounting both the trade account and the external debt with an average interest rate and performed the above (41)-(42) sustainability test.

Figure 16 shows the undiscounted and discounted time series for external asset of Euroland.

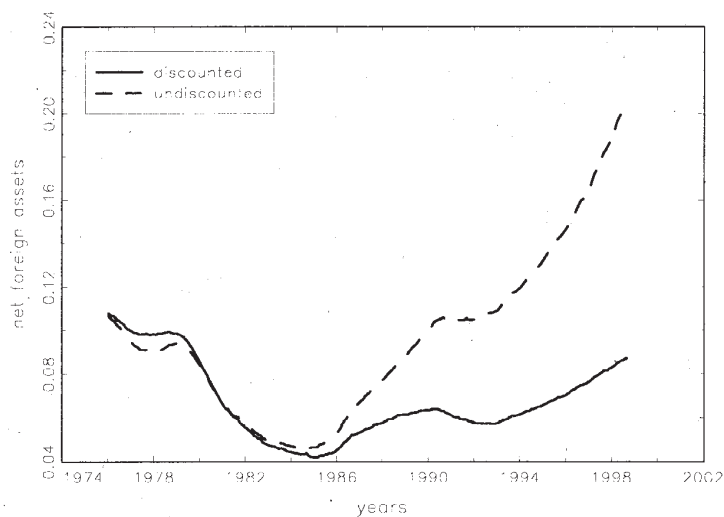


Figure 16: Undiscounted and discounted net foreign assets.

Table 2 reports test results for both types of time series for the entire time period 1978.1-1998.12.

Table 2: Sustainability Test of Euro Debt, 1978.1-1998.12

Param	undiscounted		discounted	
	Estim	t-stat	Estim	t-stat
b_1	0.00	0.00	0.00	0.0
b_2	0.76	0.05	0.53	0.04
b_3	0.45	-0.02	0.37	0.02
b_4	-0.51	-0.05	-0.06	-0.07
b_5	-0.07	-1.20	-0.002	-0.04
b_6	0.0051	0.06	-0.064	-0.88

Table 3 reports our estimation results for subperiods again for both undiscounted and discounted trade account and debt service. The results of estimation of the coefficients as to the relevance of non-sustainability of foreign assets for Euroland are not very conclusive. The coefficient b_5 , which is the relevant coefficient in our context, has the correct sign but is always insignificant.

Next we compute the estimate (41)-(42) for the two subperiods. Table 3 reports the results for undiscounted and discounted variables respectively.

Table 3: Sustainability Test of Foreign Debt for Euroland, 1978.1-1993.12 and 1994.1-1998.12

	undiscounted				discounted			
	1978.1-1993.12		1994.1-1998.12		1978.1-1993.12		1994.1-1998.12	
b_1	0.00	0.00	0.001	0.53	0.00	0.00	0.001	0.39
b_2	0.423	0.04	0.308	0.16	0.498	0.04	0.365	0.14
b_3	0.338	0.02	-0.200	-0.10	0.401	0.03	-0.266	-0.09
b_4	0.048	0.01	-0.318	-0.19	-0.062	-0.01	-0.548	-0.22
b_5	-0.042	-0.72	-0.203	-25.86	0.018	0.30	-0.061	-13.08
b_6	-0.025	-0.32	0.277	17.61	-0.086	-1.07	0.056	5.91

As can clearly be seen from the coefficients b_5 both the undiscounted as well as discounted trade time series show that there has been a rapid built-up of net foreign assets of Euroland that do not seem to be sustainable. Our tests imply there is a build-up of foreign assets that particularly occurred after the currency crisis 1992/1993.

8 Conclusions

In the paper we show that the sustainable debt in models with borrowing and lending may typically be state constrained. In order to control credit risk the lender needs to know the debt capacity of the borrower at each point in time. This knowledge seems to be necessary if one wants to move beyond an one period debt contract. We explore the problem of critical debt and creditworthiness by applying three methods. Using those method we analytically and numerically can demonstrate the region in which the borrower remains creditworthy. Imposing a ceiling of borrowing may lead to a loss of welfare if the ceiling is set to low. Moreover, in some instances it may be necessary for the borrower to first increase debt in order to decrease it. On the other hand, if the ceiling is set to high the non-explosiveness condition may not hold and creditworthiness may be lost. Moreover, we show the relation of our first method, the vector field analysis, to the Hamilton-Jacobi-Bellman (H.J.B.) equation and dynamic programming.

We discuss continuous and discrete time variants of optimal growth models with borrowing and lending and provide error bounds for the discretized version. By using our new methods we study the debt capacity of a borrower, the role of debt ceilings for lending and borrowing behavior and the effect of state dependent credit cost on the path of sustainable debt. The analytics is provided for the n capital goods model and an illustrative example is presented for the univariate case. For the latter case we also study the problem of multiple steady states which arise from nonlinear adjustment cost of investment. Our methods permits us to detect optimal and non-optimal candidates for equilibria and cut-off points for different domains of attraction. The shapes of the credit cost function - as well as the net income function - are relevant for the location of the cut-off points and thus for the result whether the economy moves to high or low level steady states.

Although the creditworthiness - curve can be computed by iterative dynamic programming it is more efficient to use the other two methods. If the credit cost is a constant factor the Hamiltonian equation can be applied. As above shown, the cut-off point and the two different domains of attraction can also be computed by using the Hamiltonian. Although the latter method is useful in computing the creditworthiness curve and sustainable debt our proposed first method appears to be more generally applicable. In any case, the computation of such creditworthiness curve serves to determine sustainable debt for any initial capital stock k_0 and thus to control credit risk for any point in time. We want to note that there are, of course, nowadays numerous empirical approaches to control for credit risk by approximating sustainable debt by empirical indicators.⁴¹ Our attempt was, however, to show how one can compute sustainable debt based on a dynamic economic model.

Finally, we would like to point out that our model can be taken to the data. For actual economies one can compute the borrowing capacity and debt ceilings by estimating the involved parameter set. We have also shown that one can compute the sustainability of debt for actual economies by using time series methods.

⁴¹In a series of papers Milesi-Ferretti and Razin (1996, 1997) have addressed the empirical issue of how to obtain proxies for measuring sustainable debt.

9 Appendices

9.1 Appendix I: The Maximum Principle and the Hamiltonian Equation

$$\text{Max}_j \int_0^\infty e^{-\theta t} f(k(t), j(t)) dt$$

$$\text{s.t. } \dot{k} = j - \sigma k$$

with:

$$f(k, (t), j(t)) = ak^\alpha - j - j^\beta k^{-\gamma}$$

$$H(k, x, j, \lambda) = \max_j H(k, x, j, \lambda)$$

$$H(k, x, j, \lambda) = \lambda f(k, j) + x(j - \sigma k)$$

$$\dot{x} = \frac{-\partial H}{\partial k} + \theta x = (\sigma + \theta)x - \lambda f_k(k, j)$$

We denote x as the co-state variable in the Hamiltonian equations and λ is equal to 1.⁴² The function $f(k, \cdot)$ is strictly concave by assumption therefore there is a function $j(k, x)$ which satisfies the first order condition of the Hamiltonian

$$f_j(k, j) + x = 0 \tag{A1}$$

$$\Rightarrow j = j(k, x) = \left(\frac{x - 1}{k^{-\gamma} \cdot \beta} \right)^{\frac{1}{1-\beta}} \tag{A2}$$

and $j(\cdot, \cdot)$ is uniquely determined thereby. It follows that (k, x) satisfy

$$\dot{k} = j(k, x) - \sigma k \tag{A3}$$

$$\dot{x} = (\sigma + \theta)x - f_k(k, j(k, x)) \tag{A4}$$

The isoclines can be obtained by the points in the (k, x) space for $\beta = 2$ where $\dot{k} = 0$ satisfies

$$x = 1 + 2\sigma k^{1-\gamma} \tag{A5}$$

and where $\dot{x} = 0$ satisfies

$$x_\pm = 1 + \vartheta k^{1-\gamma} \pm \sqrt{\vartheta^2 k^{2-2\gamma} + 2\vartheta k^{1-\gamma} - 4\alpha\gamma^{-1} k^{\alpha-\gamma}} \tag{A6}$$

⁴²For details of the computation of the equilibria in the case when one can apply the Hamiltonian, see Semmler and Sieveking (1996), appendix.

where $\vartheta = 2\gamma^{-1}(\sigma + \theta)$. Note that the latter isocline has two branches. In section 5 the steady states for certain parameter specifications are computed and the local and global dynamics studied.

9.2 Appendix II: The Discrete Time Model

Here we study more general versions of the growth model employed in section 4. We discuss a multivariate version of the model of section 4 as well as the relationship of discrete and continuous time models. This is necessary in order to obtain error bounds for discretized versions of the continuous time growth models (Semmler and Sieveking 1998a).

Suppose capital $k(t+1)$ at time $t+1$ and debt $B(t+1)$ at time $t+1$ are determined by $k(t)$ and $B(t)$ and investment rate $j(t)$ through

$$k(t+1) = g(k(t), j(t)), \quad k(0) = k \quad (\text{A7})$$

$$B(t+1) = H(k(t), B(t)) - f(k(t), j(t)), \quad B(0) = B \quad (\text{A8})$$

.. $H(k, B)$ is the interest factor which we allow to depend also on capital k , $g(k, j)$ the growth rate of capital due to investment j and $f(k, j)$ the net income rate from capital stock k and investment rate j .

More precise assumptions on g, H, f and there domains of definition will be given below. We ask, if for a given pair (k, B) is it possible to choose a sequence of investments $j(0), j(1), \dots$, in such a way that the corresponding solution $t \rightarrow (k(t), B(t))$ of (A7) and (A8) satisfies

$$\sup_{t \geq 0} B(t) < \infty$$

If so we call B subcritical for k . The supremum of all those B which are subcritical for k is denoted by $B^*(k)$. We propose to call $B^*(k)$ creditworthiness of k . The function $k \rightarrow v(k) = B^*(k)$ will be shown to satisfy the following H.J.B. equation or optimality equation.

$$H(k, v(k)) = \sup_j [f(k, j) + v(g(k, j))], \quad k \in K \quad (\text{A9})$$

Our assumptions below imply that the equation

$$H(k, C) = \sup_j [f(k, j) + v(g(k, j))]$$

has a unique solution $C = C(v, k)$ for every capital stock k and every continuous real valued function v on capital stock k . Define the operator T by

$$Tv(k) := C(v, k)$$

The assumptions stated below permit to demonstrate:

Theorem 1

(i) The H.J.B. equation (A9) admits a unique bounded continuous solution B^* , called creditworthiness.

(ii) If $v_0 = 0$ and v_n is defined recursively by $v_{n+1} = Tv_n$ then for all n , $v_n \leq v_{n+1}$ and $\lim_n v_n = B^*$.

(iii) Suppose $\inf_{k \in K} \frac{H(k, B)}{B} > c$ for some $c > 1$. Then for every solution $(k(t), B(t), j(t))$ ($t = 0, 1, 2, \dots$) if initially $B(0) > B^*(k)$ then for large t , $B(t) > c^t$; if, however, $B(0) < B^*(k)$ then $B(t) < -c^t$ for large t .

Assumptions on g, H, f

A₁: K and J are compact spaces and $g : K \times J \rightarrow \mathbb{R}$ is continuous;

A₂: $f : K \times J \rightarrow \mathbb{R}$ is continuous, $\sup_j f(k, j) \geq 0$ for all $k \in K$;

A₃: $H : \mathbb{R} \times K \rightarrow \mathbb{R}$ is continuous, $B \rightarrow H(k, B)$ is differentiable and $\frac{\partial B}{\underset{aH(k, B) > 1}{\text{undersetkinf}}} H(0, k) = 0$ for all $k \in K$.

Remark The state space K of possible capital (resource) stocks, k , as well as the space J of admissible investment rates is not bounded (or compact) in many models - such as the one treated in section 6. It makes sense however to assume that for a given initial state (k, B) and with respect to a specific problem like debt control (H.J.B. equation) there is no loss in generality to restrict k and j respectively to some compact subspace. It is plausible that with a bounded investment rate: $\|j\| \leq c$ only a bounded set of stocks k is reachable from some initial stock. In our continuous model (in section 4) Lemma 2 permits restriction to $\|j\| \leq c = \text{const}$.

Proof of theorem 1 (i) Investment $j \in J$ applied in state (k, B) produces a subsequent state

$$\{g(k, j), H(k, B) - f(k, j)\}.$$

The debt level B is subcritical for k iff for some $j \in J$

$$B^*(g(k, j)) + f(k, j) \geq H(k, B)$$

which implies

$$\sup_j [B^*(g(k, j)) + f(k, j)] \geq H(k, B^*(k))$$

If on the other hand in the above equation "i" would hold, then for some $B > B^*(k)$ and some $j \in J$

$$B^*(g(k, j)) + f(k, j) \geq H(k, B)$$

This, however, implies $B < B^*(k)$, a contradiction. Therefore, B^* satisfies the H.J.B. equation. B^* also is bounded. Let $F = \sup \{f(k, j) \mid k, j \in K \times J\}$. Since $K \times J$ is compact and f is continuous, F is finite. By assumption A_3

$$H(k, B) - f(k, j) \geq H(k, B) - F \geq cB$$

for sufficiently large B and some constant $c > 1$. Hence if $B(0)$ is large enough any solution $t \rightarrow (k(t), B(t))$ of (10), (11) satisfies

$$B(t) \geq c^t B(0)$$

which shows that B^* is bounded. We now check that T is a Lipschitz operator on the space of bounded functions $v : K \rightarrow \mathbb{R}$ with

$$\|v\| = \sup \{|v(k)| \mid k \in K\}$$

To do so let $v_1, v_2 : K \rightarrow \mathbb{R}$ be bounded and

$$\sup_j [f(k, j) + v_1(g(k, j))] \leq f(k, j(k)) + v_1(g(k, j(k))) + \varepsilon$$

for some $\varepsilon > 0$. Then

$$H(Tv_1(k), k) - H(Tv_2(k), k) \leq v_1(g(k, j(k))) - v_2(g(k, j(k))) + \varepsilon \leq \|v_1 - v_2\| + \varepsilon$$

Due to A_3 $|H(k, B_1) - H(k, B_2)| \geq \frac{1}{l}|B_1 - B_2|$ for some constant $l \in (0, 1)$ independently of k and therefore

$$Tv_1(k) - Tv_2(k) \leq l \|v_1 - v_2\| + l\varepsilon$$

Since $\varepsilon > 0$ and k was arbitrary $\|Tv_1 - Tv_2\| \leq l \|v_1 - v_2\|$. This shows that T is a Lipschitz transformation of the space of bounded functions $K \rightarrow \mathbb{R}$. Now if $v_0 = 0$, then since $\sup_j f(k, j) \geq 0$

$$v_0 = Tv_0 \geq v_0 \quad \text{and therefore}$$

$$v_{n+1} = Tv_n \geq v_n \quad \text{for all } n$$

Also, if v is continuous, then so is Tv since $K \times J$ is compact. Therefore $B^* = \lim_n v_n$ is continuous, this proves (ii).

Proof of (iii) Suppose $B > B_1 > B^*(k)$ and let $t \rightarrow (k(t), B(t), j(t))$ solve (10) and (11) with $B = B(0)$, $k = k(0)$. Compare this to the solution $t \rightarrow (k_1(t), B_1(t), j(t))$ of (10) and (11) with $k_1(0) = k$, $B_1(0) = B_1$ and the same investment.

$$\begin{aligned} B(t) - B_1(t) &\geq c(B(t-1) - B_1(t-1)) \geq \dots \geq c^t(B(0) - B_1(0)) \\ B(t) &\geq c^t(B(0) - B_1(0)) + B_1(t) \geq c^t(B(0) - B_1(0)) \end{aligned}$$

Similarly, if $B^*(k) > B_1 > B$ we find

$$B_1(t) - B(t) \geq c^t (B_1(0) - B(0))$$

and

$$-c^t (B_1(0) - B(0)) + B_1(t) \geq B(t)$$

As $B_1(\cdot)$ is bounded this proves (iii), that is

$$B(t) \geq c_1^t \text{ for large } t, \quad c_1 < c$$

in the first case and

$$B(t) \leq -c_1^t \text{ for large } t, \quad c_1 < c$$

in the second case.

9.3 Appendix III: The Continuous Time Model

Here we consider a continuous (deterministic) model of the investment process with borrowing and the corresponding function of creditworthiness. It is, however, only the approximation by a discrete model of the type of the proceeding section that will be treated here.

We first derive heuristically a H.J.B. equation which implicitly determines the creditworthiness of a capital stock. This equation in general does not admit a classical solution, i.e. is not differentiable. We propose an Euler approximation both as interpretation and as a numerical procedure. This might be useful also if the type of H.J.B. equation considered here turns up in a different context. A more detailed analysis of an example with univariate capital stock will be given in the subsequent sections.

As in the proceeding section $k(t)$ will denote the capital stock at time t :

$$k(t) \in [0, +\infty)^m$$

$B(t)$ is the level of debt accumulated at time t . There are two state equations

$$\dot{k}(t) = G(k(t), j(t)), \quad k(0) = k \tag{A10}$$

$$\dot{B}(t) = H(k(t), B(t)) - f(k(t), j(t)), \quad B(0) = B_0, \tag{A11}$$

$j(t)$ is the investment rate at time t , a control variable:

$$j(t) \in \mathbb{R}^n$$

We assume reversible investment. Precise assumptions on G , the capital growth function, H/B , the rate of interest and f , the net income rate function or production function, will be given below. What might be new for the reader is that the interest rate may also

depend on k . As before there will be a function $k \rightarrow B^*(k)$ called creditworthiness which is the supremum of all debt levels B which are subcritical for k :

$$B^*(k) = \sup \{B \mid B \text{ subcritical for } k\}$$

Here B is called subcritical for k if there is a function $t \rightarrow (k(t), B(t), j(t))$ which solves (A10) (A11) and

$$\sup_{t \geq 0} B(t) < \infty.$$

Definition: k is called an equilibrium if there is an investment j such that the corresponding solution to (A10)-(A11) with $B(0) = B^*(k)$ is constant. In other words if it is possible to simultaneously preserve the capital stock and its creditworthiness.

As is demonstrated in section 4 equilibria will be shown to satisfy equations of the form $F(k) = 0$.

The H.J.B. equation for B^* is given by

$$H(k, B^*(k)) = \text{Max}_j \left[f(k, j) + \left(\frac{d}{dk} B^*(k) \right) G(k, j) \right] \quad (\text{A12})$$

Since, in general B^* will not be (continuously) differentiable in all points B^* there is a need to explain in which sense B^* solves (A12). One way is to write

$$\left(\frac{d}{dk} B^*(k) \right) G(k, j)$$

as a directional derivative: fix $j = \text{const.}$ and let $x(t) = (k(t), B(t))$ solve (A10), (A11). Let

$$D_j B^*(k) = \left. \frac{d}{dt} B^*(k(t)) \right|_{t=0}$$

be the directional derivative of B^* at k in the direction of $\dot{x}(0)$. Then (A12) may be written as

$$H(k, B^*(k)) = \text{Max}_j [f(k, j) + D_j B^*(k)] \quad (\text{A13})$$

This form of the H.J.B. equation only requires that directional derivatives of B^* exist. We do not pursue this approach further. For a derivation of (A12) and (A13) respectively we assume that B^* is continuously differentiable and has directional derivatives respectively.

Let $(k(t), B(t), j(t))$ solve (A10), (A11) with $B = B(0) < B^*(k)$: then

$$\frac{1}{t} B(t) < \frac{1}{t} B^*(k(t))$$

for small $t > 0$ and

$$\begin{aligned} \frac{1}{t_0} H(k(s), B(s)) &\leq \frac{1}{t_0} (D_j B^* k(s) + f(k(s), j(s))) ds \\ &= \frac{1}{t_0} \frac{d}{ds} B^*(k(s)) + f(k(s), j(s)) ds \end{aligned}$$

Passing to the limit as t tends to zero we obtain

$$H(k, B) \leq \sup_j [f(k, j) + D_j B^*(k)] = \sup_j \left[f(k, j) + \left(\frac{d}{dk} B^*(k) \right) G(k, j) \right]$$

and therefore

$$H(k, B^*(k)) \leq \sup_j [f(k, j) + D_j B^*(k)] = \sup_j \left[f(k, j) + \left(\frac{d}{dk} B^*(k) \right) G(k, j) \right]$$

and similarly " \geq ".

We now define the Euler approximation, i.e. a time discrete approximation of (A10), (A11). Consider a solution

$$t \rightarrow (k(t), B(t), j(t))$$

of (A10), (A11). The function $t \rightarrow (k(t), B(t))$ may be approximated by a piecewise linear function with step size s :

$$t \rightarrow (k_s(t), B_s(t)), \quad t = 0, s, 2s, 3s, \dots$$

defined recursively by

$$(k_s(t+s), B_s(t+s)) = (k_s(t), B_s(t)) + s (G(k_s(t), j(t)), H(k_s(t), B_s(t)) - f(k_s(t), j(t)))$$

Put

$$h_s(k, B) := B + s(H(k, B))$$

$$g_s(k, j) := k + s(G(k, j))$$

Then $t \rightarrow (k_s(t), B_s(t))$ solves

$$k(t+s) = g_s(k(t), j(t)), \quad k(0) = k \tag{A14}$$

$$B(t+s) = h_s(k(t), B(t)) - s f(k(t), j(t)), \quad B(0) = B \tag{A15}$$

which is of the form (A7), (A8) of the previous section. Call the debt B (c, s)-subcritical for k if there is a solution $t \rightarrow (k(t), B(t), j(t))$ to (A14), (A15) which satisfies

$$\sup_{t \geq 0} \|j(t)\| \leq c \tag{A16}$$

and

$$\sup_{t \geq 0} B(t) < +\infty$$

Let

$$B_{c,s}^*(k) = \sup \{B \mid B(c, s) \text{ - subcritical for } k\}$$

$B_{c,s}^*(k)$ defining creditworthiness of k for the time discrete version of (A10), (A11) with step size s , when investment rate j is restricted to $\|j\| \leq c$.

The purpose of this section is to prove

Theorem 2 $B^*(k) = \sup_{c>0} \lim_{s \rightarrow 0} B_{c,s}^*(k)$.

Remark In order to obtain an algorithm to simulate $B^*(k)$, the k - space has to be discretized somehow. It is possible to derive discretization error bounds (see Falcone (1978) or lemma 4, 5 in Sieveking and Semmler 1998a). These bounds however are weak and not sufficient in practice. In general dynamic programming, i.e. iterative solution of the discrete H.J.B. equation is stable but converges very slowly.⁴³ Unfortunately $s \rightarrow B_{c,s}^*(k)$ is not monotone (i.e. increasing in s) as it stands.

Assumptions on G, H, f :

A₄: the state equation for the capital stock is $k = G(k, j) = j - \sigma(k)$ where $\sigma(k) = (\sigma_1(k), \dots, \sigma_n(k))$ and each $\sigma_i(k)$ is a function of $k = (k_1, \dots, k_n) \in [0, +\infty)^n$ which is Lipschitz continuous: $\|\sigma(k) - \sigma(k^*)\| \leq L \|k - k^*\|$ for some constant L and all $k \in [0, +\infty)^n$. In addition we assume that for every constant c there is $r > c$ such that $\sigma_i(k) \geq c$ whenever $k_i > r$ for all $i = 1, 2, \dots, n$.

A₅: the state equation for the debt is $\dot{B} = H(k, B) - f(k, j)$ where H is a continuous function $H : [0, \infty]^n \times \mathbb{R} \rightarrow \mathbb{R}$ such that $h(0, B) = 0$, $\theta_0(B - B^*) \leq H(k, B) - H(k, B^*) \leq \theta_1(B - B^*)$ for some positive $\theta_0 < \theta_1$ and all $k \in (0, +\infty)^n$, $B^*, B \in \mathbb{R}$.

$f : [0, \infty)^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous function such that for $f(k, j) = f(k, j_1, \dots, j_n)$ is a decreasing function of every $j_i (i = 1, 2, \dots, n)$ and every constant $r > 0$ there exist $L(r), c_0(r), c_1(r) > 0$ and a number $p \geq 1$ such that

$$|f(k, j) - f(k, j^*)| \leq L(r) \|j - j^*\|, \quad (\|k\| \leq r, j, j^* \in \mathbb{R}^n)$$

$$|f(k, j)| \leq c_0(r) + c_1(r) |j|_p^p, \quad (\|k\| \leq r, j \in \mathbb{R}^n)$$

Furthermore $f(k, j)$ is concave in j .

The proof of the Theorem 2 is based on the following two Lemma

Lemma 1 Suppose $B < B^*(k)$. Then there is an investment function $j(t)$ such that the corresponding solution $t \rightarrow (k(t), B(t), j(t))$ to (A10), (A11) satisfies

⁴³This results from the curse of dimension. For further treatment of the dynamic programming algorithm suggested here, see Sieveking and Semmler (1997).

$$B(T) < 0$$

for some $T > 0$.

Lemma 2 $B^*(k) = \sup_{c \geq 0} B_c^*(k) \quad (k \in (0, +\infty)^n)$

For the proofs of Lemmas 1 and 2 and Theorem 2, see Semmler and Sieveking (1998a).

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