Global warming in a basic endogenous growth model*

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Abstract

This paper studies the interrelation between anthropogenic global warming and economic growth. It is assumed that deviations from the pre-industrial average global surface temperature negatively affect aggregate output and utility. The government levies a tax on output and a tax on greenhouse gases. Assuming a basic endogenous growth model the effects of varying the tax rates are analyzed as concerns economic growth, economic welfare and as concerns the rise in average global surface temperature. Using simulations, it is demonstrated that higher emission taxes may both raise economic growth and welfare while reducing the extent of global warming. In addition, situations exist where a rise in emission taxes reduces economic growth but raises welfare. Further, the social optimum is computed and compared to the competitive economy.

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1 Introduction

According to the Intergovernmental Panel on Climate Change (IPCC) the global average surface temperature has increased by 0.6 ± 0.2 degree Celsius over the 20th century. It is very likely[1] that the 1990s was the warmest decade and 1998 the warmest year since 1861 ([8], p. 26). In addition, it is likely that statistically significant increases in heavy and extreme weather events have occurred in many mid- and high latitude areas, primarily in the Northern Hemisphere.[2] Changes in climate occur as a result of both internal variability within the climate system and external factors where the latter can be natural or anthropogenic. However, natural factors have made small contributions to the climate change observed over the past century. Instead, there is strong evidence that most of the warming observed over the last 50 years is the result of human activities. Especially, the emission of greenhouse gases (GHGs), like carbon dioxide (CO_2) or methane (CH_4) just to mention two, are considered as the cause for climate changes and theses emissions continue to alter the atmosphere in ways that are expected to affect the climate.

In the environmental economics literature there exist numerous contributions which study the interrelation between economic growth and environmental degradation (for a survey see e.g. [15] or [7]). However, as far as I know there do not exist economic studies which take climate models as starting points and then analyze the effects of global warming within a growth model. This is a bit surprising since it is probable that climate changes will also have impacts on the growth rates of economies. A great problem arising when one intends to study the economic consequences of global warming is the uncertainty as concerns the damages caused by a change of the earth climate. Nevertheless, there do exist studies which try to evaluate the costs of global warming. The IPCC estimates that a doubling of CO_2, which goes along with an increase of global average surface temperature between 1.5 and 4.5 degree Celsius, reduces world GDP by 1.5 to 2 percent (see [9],

[1] Very likely (likely) means that the level of confidence is between $90 - 99$ ($66 - 90$) percent.

[2] More climate changes are documented in [8], p. 34.

p. 218). This damage is obtained for the economy in steady state and comprises both market and nonmarket impacts. Nonmarket impacts are direct reductions of people's welfare resulting from a climate change.

But, of course, it must be repeated that there is great uncertainty in social cost estimates, especially as concerns the direct impact of climate change on individuals' utility. Usually, the direct costs of global warming are approximated by determining the willingness to pay for a benefit and the willingness to accept compensation for a cost. If there is no market the willingness to pay and the willingness to accept are estimated through surrogate markets or hypothetical markets. Surrogate markets are real markets where the environmental quality has an impact on price. Here one resorts to the hedonic property price method. For example, the price of a house may be smaller due to environmental degradation and the difference in price can be seen as the cost of the environmental pollution. The contingent valuation method resorts to hypothetical markets. In this method, people are asked to value the price of a good on a hypothetical market. In addition to the economic costs of global warming, the IPCC report also gives benefits resulting from fewer GHG emissions due to abatement activities and contrasts these with the costs assuming that the economy is on a balanced growth path. Basically, these cost benefit analyses are a summary of contributions which have been made by economists in previous works.[3] In this paper we do not intend to make cost benefit analyses. Instead, we want to analyze the effects of GHGs emissions and emission tax policies on economic growth and welfare by incorporating the greenhouse effect in a simple endogenous growth model.

The rest of the paper is organized as follows. In Section 2 we start with a description of facts concerning GHG emissions and changes in average surface temperature using a simple energy balance model (EBM). Section 3 introduces the competitive model economy. In sections 3.1 and 3.2 we present the structure of the model and we derive some analytical results for the model on the balanced growth path (BGP). In section 3.3 we present

[3] We do not go into the details of these studies. The interested reader is referred to the IPCC report (see [9]).

numerical studies which demonstrate the sensitivity of the results with respect to the parameter values. Section 4 presents and analyzes the social optimum and section 5, finally, concludes the paper.

2 Facts on GHG Emissions and the Change in Average Global Surface Temperature

Before we present the economic framework we begin with a description of scientific knowledge concerning GHG emissions and the change in global average surface temperature. The simplest method of considering the climate system of the earth is in terms of its global energy balance which is done by so-called energy balance models (EBM). According to an EBM the change in the average surface temperature on earth is described by[4]

$$\frac{dT(t)}{dt}\, c_h \equiv \dot{T}(t)\, c_h = S_E - H(t) - F_N(t),\; T(0) = T_0, \tag{1}$$

with $T(t)$ the average global surface temperature measured in Kelvin[5] (K), c_h the heat capacity[6] of the earth with dimension $J\, m^{-2}\, K^{-1}$ (Joule per square metre per Kelvin)[7] which is considered a constant parameter, S_E is the solar input, $H(t)$ is the nonradiative energy flow, and $F_N(t) = F\uparrow(t) - F\downarrow(t)$ is the difference between the outgoing radiative flux and the incoming radiative flux. S_E, $H(t)$ and $F_N(t)$ have the dimension Watt per square metre (Wm^{-2}). t is the time argument which will be omitted in the following as long as no ambiguity can arise. $F\uparrow$ follows the Stefan-Boltzmann-Gesetz which is

$$F\uparrow = \epsilon\, \sigma_T\, T^4, \tag{2}$$

[4]This subsection follows [13] chap. 10.2.1 and chap. 1. See also [6] and [4]. A more complex presentation can be found in [5].

[5]273 Kelvin are 0 degree Celsius.

[6]The heat capacity is the amount of heat that needs to be added per square metre of horizontal area to raise the surface temperature of the reservoir by 1K.

[7]1 Watt is 1 Joule per second.

with ϵ the emissivity which gives the ratio of actual emission to blackbody emission. Blackbodies are objects which emit the maximum amount of radiation and which have $\epsilon = 1$. For the earth ϵ can be set to $\epsilon = 0.95$. σ_T is the Stefan-Boltzmann constant which is given by $\sigma_T = 5.67\,10^{-8}\,Wm^{-2}K^{-4}$. Further, the ratio $F\uparrow/F\downarrow$ is given by $F\uparrow/F\downarrow = 109/88$. The difference $S_E - H$ can be written as $S_E - H = Q(1-\alpha_1)\alpha_2/4$, with $Q = 1367.5\,Wm^{-2}$ the solar constant, $\alpha_1 = 0.3$ the planetary albedo, determining how much of the incoming energy is reflected by the atmosphere and $\alpha_2 = 0.3$ that part of the energy which is absorbed by the surface of the Earth. Summarizing this discussion the EBM can be rewritten as

$$\dot{T}(t)\,c_h = \frac{1367.5}{4}\,0.21 - 0.95\,\left(5.67\,10^{-8}\right)(21/109)\,T^4,\; T(0) = T_0. \tag{3}$$

In equilibrium, i.e. for $\dot{T} = 0$, (3) gives a surface temperature of about 288.4 Kelvin which is about 15 degree Celsius. It should be mentioned that the numerical value of the heat capacity of the Earth, c_h, does not play a role as long as equilibrium solutions are considered only. Nevertheless, we will briefly discuss the numerical value of c_h. Since most of the Earth's surface is covered by seawater c_h is largely determined by the oceans. Therefore, the heat capacity of the oceans is used as a proxy for that of the earth. c_h is then given by $c_h = \rho_w\,c_w\,d\,0.7$, with ρ_w the density of seawater $(1027\,m^{-3}\,kg)$, c_w the specific heat of water $(4186\,J\,kg^{-1}\,K^{-1})$ and d the depth of the mixed layer which is set to 70 metres. The constant 0.7 results from the fact that 70 percent of the Earth are covered with seawater. Inserting the numerical values and assuming a depth of 70 metres gives $c_h = 2.10652\,10^8\,J\,m^{-2}\,K^{-1}$.

The effect of emitting GHGs is to raise the concentration of GHGs in the atmosphere which increases the greenhouse effect of the Earth. This is done by calculating the so-called radiative forcing which is a measure of the influence a GHG, like CO_2 or CH_4, has on changing the balance of incoming and outgoing energy in the Earth-atmosphere system. The dimension of the radiative forcing is Wm^{-2}. For example, for CO_2 the radiative forcing, which we denote as F, is given by

$$F = 6.3 \ln \frac{M}{M_o}\,, \tag{4}$$

with M the actual CO_2 concentration, M_o the pre-industrial CO_2 concentration and ln the natural logarithm (see [10], p. 52-53).[8] For other GHGs other formulas can be given describing their respective radiative forcing and these values can be converted in CO_2 equivalents. Incorporating (4) in (3) gives

$$\dot{T}(t)\, c_h = \frac{1367.5}{4}\, 0.21 - 0.95\, \left(5.67\, 10^{-8}\right) (21/109) T^4 + \beta_1\, (1-\xi)\, 6.3 \ln \frac{M}{M_o},\ T(0) = T_0. \tag{5}$$

β_1 is a feedback factor which captures the fact that a higher CO_2 concentration affects for example atmospheric water vapour which has effects for the surface temperature on Earth. β_1 is assumed to take values between 1.1 and 3.4. The parameter ξ, finally, captures the fact that $\xi = 0.3$ of the warmth generated by the greenhouse effect is absorbed by the oceans which transport the heat from upper layers to the deep sea. Setting $\beta_1 = 1.1$ and assuming a doubling of CO_2 implies that in equilibrium the average surface temperature rises from 288.4 to 291.7 Kelvin, implying a rise of about 3.3 degree Celsius. This is in the range of IPCC estimates[9] which yield increases between 1.5 and 4.5 degree Celsius as a consequence of a doubling CO_2 concentration ([8], p. 67).

The concentration of GHGs M evolves according to the following differential equation

$$\dot{M} = \beta_2 E - \mu M, M(0) = M_0. \tag{6}$$

E denotes emissions and μ is the inverse of the atmospheric lifetime of CO_2. As to the parameter μ we assume a value of $\mu = 0.1$.[10] β_2 captures the fact that a certain part of GHG emissions are taken up by oceans and do not enter the atmosphere. According to IPCC $\beta_2 = 0.49$ for the time period 1990 to 1999 for CO_2 emissions ([8], p. 39).

[8] The CO_2 concentration is given in parts per million (ppm).

[9] IPCC results are obtained with more sophisticated Atmosphere-Ocean General Circulation Models.

[10] The range of μ given by IPCC is $\mu \in (0.005, 0.2)$, see [8], p. 38.

3 The competitive model economy

In this section we present our economic framework. We start with the description of the structure of our economy.

3.1 The structure of the economy and analytical results

We consider an economy where one homogeneous good is produced. Further, the economy is represented by one individual with household production who maximizes a discounted stream of utility arising from per capita consumption times the number of household members subject to a budget constraint. A frequently used standard utility function U in economics is

$$U = \begin{cases} ((C\, D_1(T - T_o))^{1-\sigma} - 1)/(1-\sigma), & \text{for } \sigma > 0, \sigma \neq 1 \\ \ln C + \ln D_1(T - T_o), & \text{for } \sigma = 1. \end{cases} \tag{7}$$

C denotes per capita consumption and σ is the inverse of the intertemporal elasticity of substitution of consumption between two points in time. $D_1(T - T_o)$ gives the disutility resulting from deviations from the normal temperature[11] T_o. As to the functional form of $D_1(T - T_o)$ we assume that it is C^2 and satisfies

$$D_1(T - T_o) \begin{cases} = 1, & \text{for } T = T_o \\ < 1, & \text{for } T \neq T_o, \end{cases} \tag{8}$$

with derivative

$$\frac{\partial\, D_1(\cdot)}{\partial\, T} \equiv D_1'(\cdot) \begin{cases} > 0, & \text{for } T < T_o \\ < 0, & \text{for } T > T_o. \end{cases} \tag{9}$$

The individual's budget constraint in per capita terms is given by[12]

$$Y(1-\tau) = \dot{K} + C + A + \tau_E E L^{-1} + (\delta + n)K,\ K(0) = K_0, \tag{10}$$

[11] The normal temperature is the pre-industrial temperature.

[12] The per capita budget constraint is derived from the budget constraint with aggregate variables, denoted by the subscript g, according to $\dot{K}/K = \dot{K}_g/K_g - \dot{L}/L$.

with Y per capita production, K per capita capital, A per capita abatement activities and E emissions. $\tau \in [0, 1)$ is the tax rate on production and $\tau_E > 0$ is the tax on emission. δ is the depreciation rate of capital. L is labour which grows at rate n. In our model formulation abatement is a private good.[13] The production function is given by

$$Y = B\, K^{\alpha} \bar{K}^{1-\alpha} D_2(T - T_0), \tag{11}$$

with K per capita capital, $\alpha \in (0, 1)$ the capital share and B is a positive constant. $D_2(T - T_0)$ is the damage due to deviations from the normal temperature T_o and has the same functional form as $D_1(\cdot)$. In section 3 we will concretely specify both $D_1(\cdot)$ and $D_2(\cdot)$. $\bar{K}$ gives positive externalities from capital resulting from spillovers. This assumption implies that in equilibrium the private gross marginal returns to capital[14] are constant and equal to $\alpha B D_2(\cdot)$, thus generating sustained per capita growth if B is sufficiently large. This is the simplest endogenous growth model existing in the economics literature. However, since we are not interested in explaining sustained per capita growth but in the interrelation between global warming and economic growth this model is sufficiently elaborate.

As concerns emissions of GHGs we assume that these are a byproduct of capital used in production and expressed in CO_2 equivalents. So emissions are a function of per capita capital relative to per capita abatement activities. This implies that a higher capital stock goes along with higher emissions for a given level of abatement spending. This assumption is frequently encountered in environmental economics (see e.g. [15] or [7]). It should also be mentioned that the emission of GHGs does not affect utility and production directly but only indirectly by affecting the climate of the Earth which leads to a higher surface temperature and to more extreme weather situations. Formally, emissions are described

[13] There exist some contributions which model abatement as a public good. See e.g. [11] or [12].

[14] With gross return we mean the return to capital before tax and for the temperature equal to the pre-industrial level.

by

$$E = \left(\frac{a\,K}{A}\right)^{\gamma}, \tag{12}$$

with $\gamma > 0$ and $a > 0$ constants. The parameter a can be interpreted as a technology index describing how polluting a given technology is. For large values of a a given stock of capital (and abatement) goes along with high emissions implying a relatively polluting technology and vice versa. It should also be mentioned that our definition (12) implies that emissions are a public good, or more appropriately, a public bad. This holds because aggregate emissions E_g are given by $E_g = (aK_g/A_g)^{\gamma}$ with $K_g = L\,K$ and $A_g = L\,A$, implying $E_g = E$.

The government in our economy is modelled very simple. The government's task is to correct the market failure caused by the negative environmental externality.[15] The revenue of the government is transferred in a lump-sum manner to the household so that transfers do not affect the consumption-investment decision.

The individual's optimization problem can be written as

$$\max_{C,A} \int_0^{\infty} e^{-\rho t} L_0 e^{nt} U(C, D_1(T - T_o))dt, \tag{13}$$

subject to (7), (10), (11) and (12). ρ in (13) is the subjective discount rate, L_0 is labour supply at time $t = 0$ which we normalize to unity and which grows at constant rate n. It should be noted that in the competitive economy the individual does neither take into account the negative externality of capital, the emission of GHG, nor the positive externalities, i.e. the spillover effects.

To find the optimal solution we form the current-value Hamiltonian[16] which is

$$\begin{aligned} H(\cdot) &= ((C\,D_1(T - T_o))^{1-\sigma} - 1)/\,(1 - \sigma) + \lambda_1((1 - \tau)\,B\,K^{\alpha}\bar{K}^{1-\alpha}D_2(T - T_o) - \\ & \quad C - A - \tau_E L^{-1} a^{\gamma} K^{\gamma} A^{-\gamma} - (\delta + n)K), \end{aligned} \tag{14}$$

with λ_1 the shadow price of K. Note that we used $E = a^{\gamma}K^{\gamma}A^{-\gamma}$.

[15] How the government has to take into account the positive externality is studied in section 5.

[16] For an introduction to the optimality conditions of Pontryagin's maximum principle see [3] or [14].

The necessary optimality conditions are given by

$$\begin{aligned}
\frac{\partial H(\cdot)}{\partial C} &= C^{-\sigma} D_1^{1-\sigma} - \lambda_1 = 0, &(15)\\
\frac{\partial H(\cdot)}{\partial A} &= \tau_E L^{-1} a^{\gamma} K^{\gamma} \gamma A^{-\gamma-1} - 1 = 0, &(16)\\
\dot{\lambda}_1 &= (\rho+\delta)\lambda_1 - \lambda_1 \Big((1-\tau)\, B\, \alpha\, D_2(\cdot) - (\tau_E/LK)\, \gamma\, a^{\gamma}\, K^{\gamma} A^{-\gamma}\Big). &(17)
\end{aligned}$$

In (17) we used that in equilibrium $K = \bar{K}$ holds. Further, the limiting transversality condition $\lim_{t\to\infty} e^{-(\rho+n)t}\lambda_1 K = 0$ must hold.

Using (15) and (17) we can derive a differential equation giving the growth rate of per capita consumption. This equation is obtained as

$$\frac{\dot{C}}{C} = -\frac{\rho+\delta}{\sigma} + \frac{\alpha}{\sigma}(1-\tau)\, BD_2(\cdot) - \frac{\gamma}{\sigma}\frac{\tau_E}{LK}\, a^{\gamma} K^{\gamma} A^{-\gamma} + \frac{1-\sigma}{\sigma}\frac{D_1'(\cdot)}{D_1(\cdot)}\dot{T}, \tag{18}$$

where $D_1'(\cdot)$ stands for the derivative of $D_1(\cdot)$ with respect to T. Combining (16) and (12) yields

$$E = \left(\frac{\tau_E}{LK}\right)^{-\gamma/(1+\gamma)} a^{\gamma/(1+\gamma)}\, \gamma^{-\gamma/(1+\gamma)}. \tag{19}$$

Using (5) and (6) from section 2 with the numerical parameter values introduced and the equations derived in this section the competitive economy is completely described by the following differential equations

$$\begin{aligned}
\dot{T}\, c_h &= 71.7935 - 5.67\, 10^{-8}(19.95/109)\, T^4 + 6.3\, \beta_1\, (1-\xi)\ln\frac{M}{M_o},\ T(0) = T_0 &(20)\\
\dot{M} &= \beta_2 \left(\frac{\tau_E}{LK}\right)^{-\gamma/(1+\gamma)} a^{\gamma/(1+\gamma)}\, \gamma^{-\gamma/(1+\gamma)} - \mu M,\ M(0) = M_0 &(21)\\
\frac{\dot{C}}{C} &= -\frac{\rho+\delta}{\sigma} + \frac{\alpha}{\sigma}(1-\tau)\, BD_2(\cdot) - \frac{\gamma}{\sigma}\left(\frac{\tau_E}{LK}\right)^{1/(1+\gamma)} a^{\gamma/(1+\gamma)}\gamma^{-\gamma/(1+\gamma)} + \\
&\quad \frac{1-\sigma}{\sigma}\frac{D_1'(\cdot)}{D_1(\cdot)}\dot{T} &(22)\\
\frac{\dot{K}}{K} &= B\, D_2(T-T_0)\,(1-\tau) - \left(\frac{\tau_E}{LK}\right)^{1/(1+\gamma)} a^{\gamma/(1+\gamma)}\, \gamma^{-\gamma/(1+\gamma)}(1+\gamma) - \\
&\quad \frac{C}{K} - (\delta+n),\ K(0) = K_0, &(23)
\end{aligned}$$

where $C(0)$ can be chosen by society.

3.2 The balanced growth path

In this subsection we derive analytical results for our model economy on a balanced growth path.[17] First we define how a balanced growth path is characterized.

Definition *A balanced growth path (BGP) is a path such that* $\dot{T} = 0$, $\dot{M} = 0$ *and* $\dot{C}/C = \dot{K}/K$ *hold, with* $M \geq M_o$.

This definition contains several aspects. First, we require that the temperature and the GHG concentration must be constant along a BGP. This is a sustainability aspect. Second, the growth rate of per capita consumption equals that of per capita capital and is constant. Third, we only consider balanced growth paths with a GHG concentration which is larger than or equal to the pre-industrial level. This requirement is made for reasons of realism. Since GHG concentration has been rising monotonically over the last decades it is not necessary to consider a situation with declining GHG concentration. Proposition 1 shows that there exists a unique BGP for this economy under a slight additional assumption.

Proposition 1 *For the competitive model economy there exists a unique BGP for a constant value of* τ_E/LK *which is a saddle point with one positive and two negative real eigenvalues.*

Proof: See appendix.

This proposition shows that there exists a two-dimensional stable manifold. Solutions starting on that manifold converge to the BGP in the long run while all other solutions diverge. Since $T(0)$, $M(0)$ and $K(0)$ are given the value for $C(0)$ must be chosen such that $c(0) \equiv C(0)/K(0)$ lies on the stable manifold. Further, it cannot be excluded that the BGP goes along with a negative growth rate because the sign of the balanced growth rate depends on the concrete numerical values of the parameters. The question of whether

[17] In the following, steady state is used equivalently to balanced growth path.

there exists a BGP with a positive growth rate, i.e. a non-degenerate BGP, for a certain parameter constellation is addressed in the next section. In this section we make the assumption that a non-degenerate BGP exists.

An important aspect is that on a BGP M is constant implying that emissions of GHGs are constant, too. In a growing economy, however, this is only possible if abatement activities rise with the same rate as the capital stock. Since abatement activities are set by private agents to satisfy (16) the government has to levy the emission tax in a way such that the ratio τ_E/LK is constant. This implies that the tax on emission must rise with the same rate as the aggregate capital stock LK. However, it must be also noted that this only holds for a given relation between the ratio K/A and emissions. So, technical progress, generating a less polluting technology which implies that a given capital stock K causes less emissions, will change this outcome. In our framework this would be modelled by a decrease in a. This would affect the value of τ_E. In the following we will not go into the details of this aspect but analyze our model for constant parameter values and assume that the government keeps the ratio τ_E/LK constant.

In a next step we analyze how the balanced growth rate which we denote by $g^\star$ and which is given by (22) with $\dot{T} = 0$, reacts to changes in the production tax rate τ and to different values of the ratio τ_E/LK. To do so we differentiate $g^\star$ with respect to τ. This gives

$$\frac{\partial g^\star}{\partial \tau} = B\left(\frac{\alpha}{\sigma}\right)\left((1-\tau)D_2'(\cdot)\,\frac{\partial T^\star}{\partial \tau} - D_2(\cdot)\right),$$

with $T^\star$ denoting the surface temperature on the BGP. From (20) and (21) it is immediately seen that $\partial T^\star/\partial \tau = 0$ so that a higher tax rate on production unequivocally lowers the balanced growth rate.

This does not hold for variations of the emission tax. Differentiating $g^\star$ with respect to (τ_E/LK) yields

$$\frac{\partial g^\star}{\partial(\tau_E/LK)} = B\left(\frac{\alpha}{\sigma}\right)(1-\tau)D_2'(\cdot)\,\frac{\partial T^\star}{\partial(\tau_E/LK)} - \frac{\gamma}{\sigma(1+\gamma)}E, \tag{24}$$

with E given by (19). To get an idea about the sign of this expression we need to know the sign of $\partial T^\star/\partial(\tau_E/LK)$. $\partial T^\star/\partial(\tau_E/LK)$ is obtained by solving (20) with respect to $T^\star$, inserting $M^\star$ obtained from (21)=0 and then differentiating with respect to (τ_E/LK). It is immediately seen that this derivative is negative. Since for $T \geq T_o$, which holds on a BGP due to $M \geq M_o$, the derivative of $D_2(\cdot)$ is negative so that the first part of the above expression is positive.

With these considerations we can state that this derivative shows that an increase in the emission tax may raise or lower the balanced growth rate. This holds because there are two counteracting forces: On the one hand, a higher emission tax reduces the net marginal product of private capital, thus, reducing the balanced growth rate. On the other hand, a higher emission tax reduces the average surface temperature and, as a consequence, the damage resulting from deviations of the actual temperature from its pre-industrial level. This tends to raise the net marginal product of private capital and the incentive to invest. So we can state that a higher tax on GHG emissions may yield a double dividend, or a win-win situation, by both raising the balanced growth rate and reducing GHG emissions. It should also be mentioned that the direct negative effect of global warming as to utility does not affect this outcome. This is easily seen from (19) and (20) with $M^\star$ determined by (21)=0.

It is obvious that the specification of the damage function is most decisive as to whether our model yields a double dividend. Further, it is also feasible that an increase in the emission tax ratio lowers economic growth but, nevertheless, yields a higher welfare because a decrease in GHG emissions positively affects utility. In order to study these questions it is necessary to concretely specify the damage functions $D_1(T - T_o)$ and $D_2(T - T_o)$ and to resort to numerical examples. This is done in the next section.

3.3 Numerical examples

In this section we analyze how the growth rate and welfare react to variations in the emission tax ratio. We do this for the economy on the BGP using numerical examples and we start this subsection with a description of the parameter values we employ in our numerical analysis.

We consider one time period to comprise one year. The discount rate is set to $\rho = 0.03$, the population growth rate is assumed to be $n = 0.02$ and the depreciation rate of capital is $\delta = 0.075$. The pre-industrial level of GHGs is normalized to one, i.e. $M_o = 1$, and we set $\gamma = 1$. β_1 and ξ are set to $\beta_1 = 1.1$ and $\xi = 0.3$ (see section 2). The tax rate on output is $\tau = 0.1$ and the capital share is $\alpha = 0.45$. This value seems to be high. However, if capital is considered in a broad sense meaning that it also comprises human capital this value is reasonable.[18] B is set to $B = 0.35$ implying that the social gross marginal return to capital is 35 percent for $T = T_o$.

As to τ_E/LK we consider the values $\tau_E/LK = 0.005,\ 0.01, 0.015$. For example, in Germany the ratio of tax on mineral oil to private gross capital (excluding residential capital) was 0.0037 (0.0068) in 1999 (see [16], p. 510, 639). a is set to $a = 1.65\,10^{-4}$ so that a doubling of GHGs implies a rise in the average surface temperature of about 3.3 degree Celsius for $\tau_E/LK = 0.01$. We also consider the lower values $\tau_E/LK = 0.0005,\ 0.001, 0.0015$.[19] As to the inverse of the intertemporal elasticity of substitution of consumption we consider the values $\sigma = 1$ and $\sigma = 2$.

An important role is played by the damage functions $D_1(\cdot)$ and $D_2(\cdot)$. These will be introduced now. As to $D_1(\cdot)$ we assume the function

$$D_1(\cdot) = \left(a_1\,(T - T_o)^2 + 1\right)^{-\varphi}, \tag{25}$$

with $a_1 > 0$, $\varphi > 0$. $D_2(\cdot)$ is given by

$$D_2(\cdot) = \left(a_2\,(T - T_o)^2 + 1\right)^{-\phi}, \tag{26}$$

[18] The choice of α does not affect the qualitative results but only the magnitude of endogenous variables like the balanced growth rate for example.

[19] To get temperature increases for a doubling of GHGs in line with IPCC calculations we set $a = 1.65\,10^{-4}$ in this case.

with $a_2 > 0$, $\phi > 0$. As to the numerical values of the parameters in (25) we assume $a_1 = 0.05$ and $\varphi = 0.05$ which are left unchanged in our examples except for table 4. These values imply that a rise of the surface temperature by 1 (2, 3) degree(s) implies a decrease of utility by 0.1 (0.4, 0.8) percent for $C = 10$ with $\sigma = 1$. For $C = 100$ ($C = 1000$) the damage is about one half (one third) of that obtained for $C = 10$.[20] For $\sigma = 2$ the decrease is 0.03 (0.1, 0.2) percent for a temperature increase of 1 (2, 3) degree(s) for $C = 10$. For $C = 100$ ($C = 1000$) the damage is about one tenth (one hundredth) of that obtained for $C = 10$. Setting $a_1 = 0.1$ and $\varphi = 0.1$ gives a decrease in utility of 0.4, 1.5, 2.8 (0.1, 0.4, 0.7) percent for a temperature increase of 1, 2, 3 degrees with $\sigma = 1$ ($\sigma = 2$) for $C = 10$. In ([9]), p. 196/197, an example is given how a monetary value can be attached to a change in the risk of death as a result of a climate change. The IPCC cites the study by ([2]) who estimates this number to be on average 0.26 percent of world GDP for a doubling of CO_2 concentration[21] which corresponds in our model to a temperature increase of about 3.3 degree. But note that there are other effects which have an impact on individuals' utility, like an increase in extreme weather events for example, so that different values may be plausible, too. Further, it should also be kept in mind that empirical estimates for damages are necessarily uncertain, as already mentioned in the Introduction. This holds in particular for the direct effect of global warming as concerns people's utility.

As concerns the numerical values of a_2 and ϕ we consider the cases ($a_2 = 0.025, \phi = 0.025$), ($a_2 = 0.035, \phi = 0.035$) and ($a_2 = 0.05, \phi = 0.05$). The combination ($a_2 = 0.025, \phi = 0.025$) implies that an increase of the surface temperature by 1 (2, 3) degree(s) leads to a decrease of aggregate production by 0.06 (0.2, 0.5) percent. With ($a_2 = 0.05, \phi = 0.05$) an increase of the surface temperature by 1 (2, 3) degree(s) leads to a decrease of aggregate production by 0.2 (0.9, 1.8) percent. Comparing these values with the estimates

[20] The dependence of the damage on the level of consumption is due to the use of function (7) which, however, is frequently resorted to in economics.

[21] This holds for the economy in steady state.

published in ([9]), mentioned in the Introduction, we see that most of the values we choose tend to be within the range of that study.

In the following tables we report the results of our numerical studies. These studies compare different scenarios where it is assumed that the exogenous variables, i.e. τ_E/LK and σ, take their respective values for all $t \in [0, \infty)$ and that the economy immediately jumps to its BGP. Table 1 shows the effects of different ratios τ_E/LK where the * denotes values on the BGP.[22] $g^\star$ is the balanced growth rate given by (22) with $\dot{T} = 0$ and $W^\star$ denotes welfare. $W^\star$ is given by

$$W^\star = \int_0^\infty e^{-(\rho-n)t} U(C(t), D_1(T^\star - T_o))dt,$$

with $U(\cdot)$ given by (7) and $C(t) = C(0)e^{g^\star t}$, where we set $C(0) = 10$. We also report the steady state ratio of abatement activities to capital, $A^\star/K^\star$.

Table 1. Varying the emission tax between 0.005 and 0.015 with $a_2 = 0.025, \phi = 0.025$.

τ_E/LK	σ	$T^\star$	$M^\star$	$g^\star$	$W^\star$	$A^\star/K^\star$
0.005	1	293.3	2.8	0.0322	548.4267	$2.87\,10^{-3}$
0.005	2	293.3	2.8	0.0161	96.0148	$2.87\,10^{-3}$
0.01	1	291.7	1.99	0.0318	546.4721	$4.06\,10^{-3}$
0.01	2	291.7	1.99	0.0159	96.0572	$4.06\,10^{-3}$
0.015	1	290.7	1.6	0.0313	542.2684	$4.97\,10^{-3}$
0.015	2	290.7	1.6	0.0157	96.0557	$4.97\,10^{-3}$

This table shows that an increase in the emission tax reduces the balanced growth rate independent of the value of σ. However, the value of σ affects welfare. For $\sigma = 1$ welfare decreases as a consequence of the lower balanced growth rate. For $\sigma = 2$, welfare first rises with an increase in the emission tax rate although the balanced growth rate declines. In this case, the negative effect of a lower balanced growth rate on welfare is dominated

[22] All numerical calculations were done using Mathematica, see ([17]).

by the positive direct welfare effect resulting from a lower increase in the average surface temperature. If the emission tax rate is further increased welfare declines again. The effect that an increase in the emission tax rate lowers the balanced growth rate but raises welfare is the more likely the larger the direct effect of a temperature increase on utility. This is demonstrated below in table 4 where we set $a_1 = 0.01$ and $\varphi = 0.01$. In the next table we consider the case $(a_2 = 0.05, \phi = 0.05)$.

Table 2. Varying the emission tax between 0.005 and 0.015 with $a_2 = 0.05, \phi = 0.05$.

τ_E/LK	σ	$T^\star$	$M^\star$	$g^\star$	$W^\star$	$A^\star/K^\star$
0.005	1	293.3	2.8	0.0284	510.0262	$2.87\,10^{-3}$
0.005	2	293.3	2.8	0.0142	95.6984	$2.87\,10^{-3}$
0.01	1	291.7	1.99	0.0296	524.5288	$4.06\,10^{-3}$
0.01	2	291.7	1.99	0.0148	95.8829	$4.06\,10^{-3}$
0.015	1	290.7	1.6	0.03	529.777	$4.97\,10^{-3}$
0.015	2	290.7	1.6	0.015	95.9573	$4.97\,10^{-3}$

Table 2 shows that an increase in the damage caused by a higher average surface temperature yields a double dividend.[23] In this case, a rise in the emission tax yields both a higher balanced growth rate and a lower rise in the average global surface temperature independent of the intertemporal elasticity of substitution of consumption. Of course, this implies that welfare along the BGP unequivocally rises.

We do not show a table for the case ($a_2 = 0.035$, $\phi = 0.035$). The results for this case are in between the results we obtained in table 1 and table 2. This means that rising τ_E/LK from 0.005 to 0.01 first rises the balanced growth rate. Increasing τ_E/LK further from 0.01 to 0.015 then reduces the balanced growth rate which, however, remains larger than for $\tau_E/LK = 0.005$. This holds for both $\sigma = 1$ and $\sigma = 2$. The same holds as

[23]It should be recalled that this effect is independent of the function $D_1(\cdot)$, which determines the direct utility effect of global warming.

concerns welfare, i.e. welfare first rises with an increase in τ_E/LK and then declines but remains higher than for the initial level of τ_E/LK.

Next we consider the case of smaller values for the emission tax ratio. The results for $(a_2 = 0.025, \phi = 0.025)$ are shown in table 3.

Table 3. Varying the emission tax between 0.0005 and 0.0015 with $a_2 = 0.025, \phi = 0.025$.

τ_E/LK	σ	$T^\star$	$M^\star$	$g^\star$	$W^\star$	$A^\star/K^\star$
0.0005	1	293.3	2.8	0.0348	574.2773	$2.87\,10^{-4}$
0.0005	2	293.3	2.8	0.0174	96.2028	$2.87\,10^{-4}$
0.001	1	291.7	1.99	0.0355	583.0303	$4.06\,10^{-4}$
0.001	2	291.7	1.99	0.0177	96.3169	$4.06\,10^{-4}$
0.0015	1	290.7	1.6	0.0358	587.0428	$4.97\,10^{-4}$
0.0015	2	290.7	1.6	0.0179	96.3722	$4.97\,10^{-4}$

Table 3 shows that with smaller values for the emission tax ratio a double dividend is obtained even in the case where the damage caused by a climate change is relatively small. Thus, the smaller the emission tax ratio the more likely it is that raising this policy variable both raises the long run growth rate and reduces the increase in the average surface temperature. It should be recalled that smaller values of τ_E/LK are only feasible for economies with a smaller value of a, i.e. in economies with a less polluting technology. The reason for the outcome in table 3 is that the negative growth effect of an increase in the emission tax is smaller if the technology in use is less polluting. This is seen from (24) together with (19). From an economic point of view the interpretation is as follows. If the emission tax rate is increased the individual shifts resources from investment to abatement. The rise in abatement relative to the capital stock in order to get a certain decrease in emissions, however, is smaller when the technology in use is relatively clean.[24] Consequently, the negative growth effect is smaller compared to a

[24]This is seen by differentiating A/K, obtained from (16), with respect to τ_E/LK.

situation with a more polluting technology. Of course, the double dividend is also obtained for ($a_2 = 0.035, \phi = 0.035$) and ($a_2 = 0.05, \phi = 0.05$). In the latter cases, the effects are larger in magnitude compared to those in table 3.

In table 4 we show that with a higher direct damage of a rise in temperature as concerns utility, an increase in the emission tax reduces the balanced growth rate but leads to higher welfare. This holds for $\sigma = 2$ but for $\sigma = 1$ only when τ_E/LK is increased from 0.005 to 0.01. In the latter case, raising τ_E/LK further reduces welfare. Together with table 1, this shows that a rise in welfare going along with a smaller growth rate is the more likely the smaller the intertemporal elasticity of substitution of consumption $1/\sigma$. The reason for that outcome is that with a smaller intertemporal elasticity of substitution the individual is less willing to shift utility benefits into the future. Therefore, he prefers a higher utility today, resulting from a temperature decrease, to a higher growth rate of consumption which would yield utility only in the future.

Table 4. Varying the emission tax between 0.005 and 0.015 with $a_2 = 0.025, \phi = 0.025$ and $a_1 = 0.1, \varphi = 0.1$.

τ_E/LK	σ	$T^\star$	$M^\star$	$g^\star$	$W^\star$	$A^\star/K^\star$
0.005	1	293.3	2.8	0.0322	540.1017	$2.87\,10^{-3}$
0.005	2	293.3	2.8	0.0161	95.6688	$2.87\,10^{-3}$
0.01	1	291.7	1.99	0.0318	541.2841	$4.06\,10^{-3}$
0.01	2	291.7	1.99	0.0159	95.8472	$4.06\,10^{-3}$
0.015	1	290.7	1.6	0.0313	539.1154	$4.97\,10^{-3}$
0.015	2	290.7	1.6	0.0157	95.9293	$4.97\,10^{-3}$

In next section we present and analyze the social optimum and compare the outcome with the competitive model economy.

4 The social optimum

In formulating the optimization problem, a social planner takes into account both the positive and negative externalities of capital. Consequently, for the social planner the resource constraint is given by

$$\dot{K} = B\,K\,D_2(T - T_o) - C - A - (\delta + n)K, K(0) = K_0. \tag{27}$$

The optimization problem is[25]

$$\max_{C,A} \int_0^\infty e^{-\rho t} L_0 e^{nt} \left(\ln C + \ln D_1(T - T_o)\right) dt, \tag{28}$$

subject to (27), (5), (6) and (12), where $D_1(\cdot)$ and $D_2(\cdot)$ are again given by (25) and (26).

To find necessary optimality conditions we formulate the current-value Hamiltonian which is

$$\begin{aligned} H(\cdot) \;\; = \;\; & (\ln C + \ln D_1(T - T_o)) + \lambda_2(B\,K D_2(T - T_o) - C - A - (\delta + n)K) + \\ & \lambda_3 \left(\beta_2\, a^\gamma K^\gamma A^{-\gamma} - \mu M\right) + \lambda_4\,(c_h)^{-1} \cdot \\ & \left(\frac{1367.5}{4}\,0.21 - \left(5.67\,10^{-8}\right)(19.95/109)T^4 + \beta_1\,(1 - \xi)\,6.3 \ln \frac{M}{M_o}\right), \end{aligned} \tag{29}$$

with λ_i, $i = 2, 3, 4$, the shadow prices of K, M and T respectively and $E = a^\gamma K^\gamma A^{-\gamma}$. Note that λ_2 is positive while λ_3 and λ_4 are negative.

The necessary optimality conditions are obtained as

$$\frac{\partial H(\cdot)}{\partial C} = C^{-1} - \lambda_2 = 0, \tag{30}$$

$$\frac{\partial H(\cdot)}{\partial A} = -\lambda_3\,\beta_2\,a^\gamma\,K^\gamma\,\gamma A^{-\gamma-1} - \lambda_2 = 0, \tag{31}$$

$$\dot{\lambda}_2 = (\rho + \delta)\,\lambda_2 - \lambda_2\,B\,D_2(\cdot) - \lambda_3\,\beta_2\,\gamma\,a^\gamma\,K^{\gamma-1}\,A^{-\gamma}, \tag{32}$$

$$\dot{\lambda}_3 = (\rho - n)\,\lambda_3 + \lambda_3\,\mu - \lambda_4\,(1 - \xi)\,\beta_1\,6.3\,c_h^{-1}\,M^{-1}, \tag{33}$$

$$\dot{\lambda}_4 = (\rho - n)\,\lambda_4 - \frac{D_1'(\cdot)}{D_1} - \lambda_2\,B\,K\,D_2'(\cdot) + \frac{\lambda_4\,(5.67\,10^{-8}(19.95/109)\,4\,T^3)}{c_h}. \tag{34}$$

[25] For the social optimum we only study the case $\sigma = 1$.

Further, the limiting transversality condition $\lim_{t\to\infty} e^{-(\rho+n)t}(\lambda_2 K + \lambda_3 T + \lambda_4 M) = 0$ must hold.

Comparing the optimality conditions of the competitive economy with that of the social planner demonstrates how the government has to set taxes in order to replicate the social optimum. Proposition 2 gives the result.

Proposition 2 *The competitive model economy replicates the social optimum if τ_E/LK and τ are set according to*

$$\frac{\tau_E}{LK} = \beta_2 \frac{-\lambda_3}{\lambda_2 K}\ , \quad \tau = 1 - \alpha^{-1}.$$

Proof: The first condition is obtained by setting (16)=(31). The second is obtained by setting the growth rate of per capita consumption in the competitive economy equal to that of the social optimum. □

This proposition shows that the emission tax per aggregate capital has to be set such that it equals the effective price of emissions, $-\lambda_3\beta_2$, divided by the shadow price of capital times per capita capital, $\lambda_2 K$, for all $t \in [0, \infty)$. This makes the representative household internalize the negative externality associated with capital. Further, it can be seen that, as usual, the government has to pay an investment subsidy (or negative production tax) of $\tau = 1 - \alpha^{-1}$. The latter is to make the representative individual take into account the positive spillover effects of capital. The subsidy is financed by the revenue of the emission tax and/or by a non-distortionary tax, like a consumption tax, or a lump-sum tax.

From (30) and (31) we get

$$\frac{A}{K} = \left(c\,(-\lambda_3)\,\beta_2\,\gamma\,a^{\gamma}\right)^{1/(1+\gamma)}\ , \tag{35}$$

with $c \equiv C/K$. Using (35), (30) and (32) the social optimum is completely described by the following system of autonomous differential equations

$$\dot{C} \;=\; C\left(B\,D_2(\cdot) - (\rho+\delta) - ((C/K)\,(-\lambda_3)\,\beta_2\,\gamma\,a^{\gamma})^{1/(1+\gamma)}\right), \tag{36}$$

$$
\begin{aligned}
\dot{K} &= K\left(B\,D_2(\cdot) - \frac{C}{K} - ((C/K)\,(-\lambda_3)\,\beta_2\,\gamma\,a^{\gamma})^{1/(1+\gamma)} - (\delta+n)\right),\ K(0) = K_0, &(37)\\
\dot{M} &= (C/K)^{-\gamma/(1+\gamma)}\,(-\lambda_3)^{-\gamma/(1+\gamma)}\,\beta_2^{1/(1+\gamma)}\,\gamma^{-\gamma/(1+\gamma)}\,a^{\gamma/(1+\gamma)} - \mu\,M, M(0){=}M_0, &(38)\\
\dot{T} &= c_h^{-1}\Big(71.7935 - 5.67\,10^{-8}(19.95/109)T^4 + 6.3\beta_1\,(1-\xi)\ln\frac{M}{M_o}\Big), T(0) = T_0, &(39)\\
\dot{\lambda}_3 &= (\rho - n)\,\lambda_3 + \lambda_3\,\mu - \lambda_4\,(1-\xi)\,\beta_1\,6.3\,c_h^{-1}\,M^{-1}, &(40)\\
\dot{\lambda}_4 &= (\rho - n)\,\lambda_4 - \frac{D_1'(\cdot)}{D_1} - B\,\frac{K}{C}\,D_2'(\cdot) + \lambda_4\left(5.67\,10^{-8}(19.95/109)\,c_h^{-1}4\,T^3\right). &(41)
\end{aligned}
$$

As for the competitive economy a BGP is given for variables $T^\star$, $M^\star$, $\lambda_3^\star$, $\lambda_4^\star$ and $c^\star$ such that $\dot{T} = \dot{M} = 0$ and $\dot{C}/C = \dot{K}/K$ holds, with $M \geq M_o$. It should be noted that $\dot{T} = \dot{M} = 0$ implies $\dot{\lambda}_3 = \dot{\lambda}_4 = 0$. Proposition 3 gives sufficient conditions for a unique BGP to exist in the social optimum.

Proposition 3 *Assume that* $D_i''(\cdot) < 0$, $i = 1, 2$, *and* $\gamma < (4/3)\,K(1)\,T_o^4$, *with* $K(1) = 5.67\,10^{-8}(19.95/109)/(6.3\beta_1\,(1-\xi))$. *Then there exists a unique BGP for the social optimum.*

Proof: See appendix.

It should be noted, that this proposition gives conditions which are sufficient but not necessary for a unique BGP so that a unique BGP may exist even if they are not fulfilled. The first condition states that the damage functions are strictly concave. The second gives a condition as to the structural parameter γ which determines the level of emissions in the economy. Inserting $\beta_1 = 1.1$, $\xi = 0.3$ (see section 2) and $T_o = 288.4$ gives $\gamma < 19.73$ which does not impose a severe limitation. For our numerical examples the existence of a unique BGP is always assured in the social optimum.

As to the local dynamics it is difficult to derive concrete results for the analytical model. This is due, among other things, to the determinant of the Jacobian matrix at the steady state which may be positive or negative. Nevertheless, something can also be said with respect to the analytical model. This is the contents of proposition 4.

Proposition 4 *Assume there exists a unique BGP for the social optimum. Then there* *exists at most a two-dimensional stable manifold. Further, a Hopf bifurcation can be excluded.*

Proof: See appendix.

This proposition shows that the BGP in the social optimum is a saddle point and a Hopf bifurcation generating persistent limit cycles are not possible. However, we cannot answer whether the eigenvalues are real or complex conjugate. Therefore, in order to gain further insight in the structure of the social optimum, we compute steady state values[26] and the eigenvalues of the Jacobi matrix at the steady state for the numerical examples in section 3.2. The results for the case $a = 1.65\,10^{-3}$ and $a_1 = 0.05$ and $\varphi = 0.05$ are show in table 5.

Table 5. Steady state values and eigenvalues in the social optimum for $a = 1.65\,10^{-3}$, $a_1 = 0.05$ and $\varphi = 0.05$.

a_2	ϕ	$T^\star$	$M^\star$	$A^\star/K^\star$	Eigenvalues
0.025	0.025	291.1	1.7	$4.65\,10^{-3}$	0.005 ± 6.84149, 0.005 ± 0.108954
0.05	0.05	289.3	1.2	$6.73\,10^{-3}$	0.005 ± 6.71623, 0.005 ± 0.180977

It can be seen that the higher the damage caused by a temperature increase with respect to aggregate output the smaller the optimal increase in GHG emissions and in the average global surface temperature. It should be noted that the smaller increase in GHGs is due to higher abatement spending relative to the capital stock. The same outcome is obtained when the direct damage as concerns utility, caused by higher surface temperature, is set higher. This is seen in table 6 where we set $a_1 = 0.1$ and $\varphi = 0.1$.

[26]The steady state values were computed using Newton's method.

Table 6. Steady state values and eigenvalues in the social optimum

for $a = 1.65\,10^{-3}$, $a_1 = 0.1$ and $\varphi = 0.1$.

a_2	ϕ	$T^\star$	$M^\star$	$A^\star/K^\star$	Eigenvalues
0.025	0.025	290.7	1.6	$5.06\,10^{-3}$	0.005 ± 6.81329, 0.005 ± 0.113433
0.05	0.05	289.2	1.19	$6.82\,10^{-3}$	0.005 ± 6.71187, 0.005 ± 0.186833

In the last table, finally, we look at the optimal steady state values of GHG emissions and of the temperature in an economy with a less polluting technology, i.e. with a lower value of a. It is seen that the ratio of abatement to capital is higher compared to the situation with a more polluting technology shown in table 5 and, consequently, the optimal values of GHG emissions and of the temperature in steady state are lower.

Table 7. Steady state values and eigenvalues in the social optimum

for $a = 1.65\,10^{-4}$, $a_1 = 0.05$ and $\varphi = 0.05$.

a_2	ϕ	$T^\star$	$M^\star$	$A^\star/K^\star$	Eigenvalues
0.025	0.025	288.8	1.07	$7.52\,10^{-4}$	0.005 ± 6.67527, 0.005 ± 0.284293
0.05	0.05	288.5	1.02	$7.92\,10^{-4}$	0.005 ± 6.6432, 0.005 ± 0.524543

5 Conclusions

In this paper we have studied the interrelation between anthropogenic global warming and economic growth. We analyzed both the competitive economy and the social optimum and derived tax rates which make the competitive economy replicate the social optimum. Further, using simulations we could derive the following results:

1. A situation may be observed where an increase in the emission tax reduces the temperature increase and raises both economic growth and welfare. Such a double dividend is the more likely the higher the damage caused by the increase in temperature

concerning aggregate output and the smaller the initial level of the emission tax rate. It should be noted that the level of the emission tax rate will be the smaller the less polluting the technology employed. Consequently, economies with a cleaner technology are more likely to experience a double dividend when the emission tax is raised in order to reduce atmospheric GHG concentrations.

2. A situation can be observed where an increase in the emission tax rate reduces the balanced growth rate but leads to higher welfare. This outcome is the more likely the higher the direct negative effect of a temperature increase on utility. Further, for a given damage function this effect is the more likely, the smaller the intertemporal elasticity of substitution of consumption.

3. In the social optimum the increase in temperature is the smaller the higher the damage caused by the temperature increase concerning aggregate output and utility. Further, optimal spending for abatement is the higher and, consequently, the increase in temperature is the smaller the less polluting the technology in use is.

The results in this paper were derived in a model with a tax on production. If the tax on production is replaced by an income tax a double dividend is more likely. This holds because then the individual can subtract its spending for abatement from its tax payments so that the negative growth effect of a rise in the emission tax ratio is smaller compared to the model with a tax on production.

A last point which should be mentioned is that the results derived hold on average for the world economy. In reality, however, the increase in temperature on earth will not be the same for all regions. So, there are regions where the increase in average temperature is much higher than in other regions. The same holds for the damages caused by a climate change. To take into account this aspect a model has to be constructed where the temperature increase is described by a partial differential equation which depends on time and on the relative position on earth and where the damage also depends on both the temperature and on the region under consideration.

Appendix

Proof of proposition 1. To show uniqueness of the steady state we solve (20)=0, (21)=0 and (22)=(23) with respect to T, M and $c \equiv C/K$. Setting (20)=0 gives

$$\begin{aligned} T_{1,2} &= \pm 99.0775\,(71.7935 + 6.3\,\beta_1\,(1-\xi)\,\ln(M/M_o))^{1/4} \\ T_{3,4} &= \pm 99.0775\sqrt{-1}\,(71.7935 + 6.3\,\beta_1\,(1-\xi)\,\ln(M/M_o))^{1/4}. \end{aligned}$$

Clearly $T_{3,4}$ are not feasible. Further, since $M \geq M_o$ only the positive solution of $T_{1,2}$ is feasible. Uniqueness of M and c on the BGP is immediately seen.[27] To study the local dynamics we note that the economy around the BGP is described by (20), (21) and $\dot{c} = c\,(\dot{C}/C - \dot{K}/K)$. The Jacobian matrix J corresponding to this dynamic system is obtained as

$$J = \begin{pmatrix} -4\,(5.67\,10^{-8})\,(19.95/109)\,(T^\star)^3\,c_h^{-1} & 6.3\,\beta_1\,(1-\xi)\,(M^\star)^{-1}\,c_h^{-1} & 0 \\ 0 & -\mu & 0 \\ a_{31} & \frac{1-\sigma}{\sigma}\,\frac{D_1'(\cdot)}{D_1}\,c^\star\,6.3\,\beta_1\,(1-\xi)\,(M^\star)^{-1}\,c_h^{-1} & c^\star \end{pmatrix},$$

with * denoting steady state values and

$$a_{31} = c^\star \left((1-\tau)\,B\,D_2'(\cdot)\left(\frac{\alpha}{\sigma} - 1\right) + \frac{1-\sigma}{\sigma}\left(\frac{D_1''(\cdot)\,D_1(\cdot) - D_1'(\cdot)^2}{D_1(\cdot)^2}\,\dot{T} + \frac{\partial \dot{T}}{\partial T}\,\frac{D_1'(\cdot)}{D_1(\cdot)}\right)\right).$$

The eigenvalues of J are given by

$$e_1 = -4(5.67\,10^{-8})(19.95/109)\,(T^\star)^3\,c_h^{-1}, \quad e_2 = -\mu \quad \text{and} \quad e_3 = c^\star.$$

Thus proposition 1 is proved. □

Proof of proposition 3. To prove proposition 3 we compute $M^\star$ from (38)=0 as

$$M^\star = (-\lambda_3)^{-\gamma/(1+\gamma)}\,(c^\star)^{-\gamma/(1+\gamma)}\,\beta_2^{1/(1+\gamma)}\,\gamma^{-\gamma/(1+\gamma)}\,a^{\gamma/(1+\gamma)}/\mu\,,$$

with $c^\star = \rho - n$ from $\dot{C}/C = \dot{K}/K$. Inserting $M^\star$ in $\dot{\lambda}_3$ and solving $\dot{\lambda}_3 = 0$ with respect to λ_4 yields

$$\lambda_4^\star = -K(2)(-\lambda_3)^{1/(1+\gamma)},$$

[27] We neglect the economically meaningless steady state $c^\star = 0$.

with

$$K(2) = \frac{\rho - n + \mu}{(1-\xi)\,\beta_1\, 6.3\, c_h^{-1}\, (c^\star)^{\gamma/(1+\gamma)}\, \beta_2^{-1/(1+\gamma)}\, \gamma^{\gamma/(1+\gamma)}\ a^{-\gamma/(1+\gamma)}\, \mu} > 0\,.$$

Inserting $\lambda_4^\star$ and $M^\star$ in $\dot\lambda_4$ and $\dot T$ respectively and setting the latter two equations equal to zero leads to

$$(-\lambda_3)^{1/(1+\gamma)} = \frac{-D_1'/D_1 - D_2'\, B/c}{K(3) + K(4)\, T^3}\,, \tag{42}$$

$$(-\lambda_3)^{1/(1+\gamma)} = \left(\frac{K(5)}{e^{(-71.7935+5.67\,10^{-8}(19.95/109)T^4)/(6.3\,\beta_1\,(1-\xi))}}\right)^{1/\gamma}\,, \tag{43}$$

with

$$\begin{aligned} K(3) &= (\rho - n)\, K(2) > 0\,,\\ K(4) &= 5.67\,10^{-8}(19.95/109)\, c_h^{-1}\, 4\, K(2) > 0\,,\\ K(5) &= c^{-\gamma/(1+\gamma)}\, \beta_2^{1/(1+\gamma)}\, \gamma^{-\gamma/(1+\gamma)}\, a^{\gamma/(1+\gamma)}/(\mu\, M_0) > 0\,. \end{aligned}$$

Setting the r.h.s. in (42) equal to the r.h.s. in (43) gives

$$\frac{-D_1'/D_1 - D_2'\, B/c}{K(5)^{1/\gamma}} = \frac{K(3) + K(4)\, T^3}{e^{(K(6)+K(1)T^4)/\gamma}}\,, \tag{44}$$

with

$$\begin{aligned} K(6) &= -71.7935/(6.3\,\beta_1\ (1-\xi))\,,\\ K(1) &= 5.67\,10^{-8}(19.95/109)/(6.3\,\beta_1\ (1-\xi))\,. \end{aligned}$$

For $T = T_o$ the r.h.s. of (44) is zero and the l.h.s. is strictly positive. Further, the derivative of the l.h.s. of (44) is strictly positive for $D_i''(\cdot) < 0$, $i = 1, 2$. The derivative of the r.h.s. is given by

$$e^{(-K(6)-K(1)T^4)/\gamma}\, 3\, T^2\, K(4) \left(1 - \frac{4\, K(3)\, K(1)\, T}{3\,\gamma\, K(4)} - \frac{4}{3\,\gamma}\, K(1)\, T^4\right)\,.$$

Using $K(1) = 5.67\,10^{-8}(19.95/109)/(6.3\,\beta_1\ (1-\xi))$ shows that $\gamma < (4/3)\, K(1)\, T_o^4$ is a sufficient condition for the derivative to be negative for $T \geq T_o$. Thus, proposition 3 is proved. □

Proof of proposition 4. To prove proposition 4 we first recall that the dynamics around the BGP are described by the dynamic system consisting of the equations $\dot{c}/c = \dot{C}/C - \dot{K}/K = n - \rho + c$, (38), (39), (40) and (41). Since $c^\star = 0$ can be excluded because c is raised to a negative power in (38), we consider the differential equation for c in the rate of growth. Equation $\dot{c}/c$ shows that $C(0)$ must be chosen such that c takes its steady state value at $t = 0$ meaning that c is a constant. The dynamics of the other variables then are described by (38), (39), (40) and (41). The Jacobian matrix J is given by

$$J = \begin{pmatrix} -\mu & 0 & \frac{a^{\frac{\gamma}{1+\gamma}}\,\beta_2^{\frac{1}{1+\gamma}}\,\gamma^{1-\frac{\gamma}{1+\gamma}}\,(-\lambda_3{}^\star)^{-\frac{\gamma}{1+\gamma}}}{(-\lambda_3{}^\star)\,(c^\star)^{\frac{\gamma}{1+\gamma}}\,(1+\gamma)} & 0 \\ \frac{6.3\,\beta_1\,(1-\xi)}{c_h\,M^\star} & \frac{5.67\,10^{-8}\cdot 19.95\cdot 4\,(T^\star)^3}{(-1)\,109\,c_h} & 0 & 0 \\ \frac{6.3\,\beta_1\,(1-\xi)\,\lambda_4^\star}{c_h\,(M^\star)^2} & 0 & \mu - n + \rho & \frac{-6.3\,\beta_1\,(1-\xi)}{c_h\,M^\star} \\ 0 & a_{42} & 0 & -n + \rho + \frac{5.67\,10^{-8}\cdot 19.95\cdot 4\,(T^\star)^3}{109\,c_h} \end{pmatrix},$$

with

$$\begin{aligned} a_{42} \;=\; & \frac{12\cdot 5.67\,10^{-8}\cdot 19.95\,\lambda_4^\star\,(T^\star)^2}{109\,c_h} + \frac{2\,a_1\,\varphi}{1 + a_1\,\left(T^\star - T_o\right)^2} + \\ & \frac{2\,a_2\,\varphi\,B\,\left(1 + a_2\,\left(T^\star - T_o\right)^2\right)^{-1-\phi}}{c^\star} - \frac{4\,a_1^2\,\varphi\,\left(T^\star - T_o\right)^2}{\left(1 + a_1\,\left(T^\star - T_o\right)^2\right)^2} + \\ & \frac{4\,a_2^2\,(-1-\phi)\,\phi\,B\,\left(1 + a_2\,\left(T^\star - T_o\right)^2\right)^{-2-\phi}\,\left(T^\star - T_o\right)^2}{c^\star}. \end{aligned}$$

The eigenvalues of J are calculated according to

$$e_{1,2,3,4} = \frac{\rho - n}{2} \pm \sqrt{\left(\frac{\rho - n}{2}\right)^2 - \frac{K_1}{2} \pm \sqrt{\left(\frac{K_1}{2}\right)^2 - \det J_1}},$$

with K_1 defined as

$$K_1 = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} + 2 \begin{vmatrix} a_{12} & a_{14} \\ a_{32} & a_{34} \end{vmatrix}$$

with a_{ij} the element of the i-th column and j-th row of the matrix J (see [1]). It is immediately seen that $K_1 < 0$ holds in our model. In [1] it is shown that $K_1 > 0$ is a necessary condition for a Hopf bifurcation which leads to limit cycles. Consequently, a Hopf bifurcation can be excluded in the social optimum. Further, the eigenvalues are symmetrical around $(\rho - n)/2$ so that there are at most two negative eigenvalues or two eigenvalues with negative real parts. This depends on the signs of $\det J$ and of K_1. However, we cannot derive more concrete results for our analytical model. □

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