Multiple Equilibria, History Dependence, and Global Dynamics in Intertemporal Optimization Models

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Abstract

Multiple equilibria and history dependent optimal solutions are important features of a wealth of widely diverse economic models. These features are typically related to the presence of market imperfections, expectational phenomena, and the like. Less known is that they can also arise in efficient deterministic intertemporal optimization models. We examine different mechanisms that can generate multiple optimal equilibria in models of the latter type, and discuss the properties of the thresholds that separate the basins of attraction of the different equilibria. As most of the existing literature, the paper focuses on one-dimensional state space models. However, an extension to the two-dimensional case is also presented. Since in many important instances the thresholds cannot be found analytically, we present three methods that allow to compute and analyze them numerically. Finally, we give a cursory review of efficient dynamic economic models with multiple equilibria.

Keywords: Dynamic optimization models, Multiple equilibria, History dependence, Thresholds, Skiba points, Numerical methods.

JEL classification: C61, C62

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1 Introduction

A prominent figure of medieval philosophy, Buridan's donkey, is shown standing at equal distance from two identical and equidistant bales of hay, unable to decide towards which bale to go.[1] In modern economic parlance, one would say that Buridan's donkey is a rational economic agent indifferent between moving towards the one or the other bale. Its current position is a decision threshold – any small movement away from it will destroy the indifference and motivate a unique optimal course of action: moving to the closest bale. The two bales can be termed the donkey's two optimal long-run stationary solutions.[2]

Buridan's story provides an overly simplified example of a situation where there is indifference between two or more optimal courses of action. Less trivial occurrences are possible in more complex environments. Consider, for example, the following case. Our donkey, that prefers more hay to less, is hesitating between a large bale close by, and a small one further away. Does its indifference necessarily reflect any kind of irrationality? Not so, if one takes into account the disutility associated with going to go to either bale – that is, the costs associated with the optimal trajectory towards the two long-term optimal solutions. If the road towards the large bale, although short, is steep and bumpy, the donkey may be well advised to follow the longer but easier way leading to the smaller one. Simply stated, the best course of action in a given situation typically does not depend exclusively upon the final rewards, that may be widely different, but also upon the costs and benefits en route towards these rewards.

At the threshold separating two optimal courses of actions the donkey may strictly prefer going to the one or to the other hay bale rather than remaining immobile. Alternatively, if the stationary utility at the threshold is sufficiently high and the road leading to the hay bales sufficiently arduous, it may be indifferent between going and staying. In both cases, however, the threshold is unstable – all optimal trajectory lead away from it towards one or the other long-term equilibria. Thus, the economic consequences are similar. Among them one should note, most importantly, history dependence: In the presence of a threshold, the optimal long-run stationary solution toward which an optimally controlled system converges does depend upon the initial conditions. A

[1] For details on the life and the work of Jean Buridan – undoubtedly one of the most influential philosophers and logicians of the late Middle Ages – see Faral (1949) or Michael (1985).

[2] In other words, the bales are attractors for a rational, i.e. optimizing donkey.

rational donkey born sufficiently close to it will spend its life at the large hay bale. The same donkey, born further away, will not. Economies or economic systems that are identical up to slightly different initial conditions may look very different on the long term. More precisely, the optimal long-run solution toward which an optimally controlled system is attracted may depend on the initial conditions – that is more often than not, on chance rather than wisdom.

In practical policy-making, it may be crucial to recognize whether or not a given problem has multiple stable optimal equilibria and, if so, to precisely locate thresholds separating the basins of attraction towards these equilibria. Consider again our donkey, but assume now that it is on a float going down a river. Two bales of hay are on the shore, both downstream of the donkey's initial position, with the largest one the higher one on the river. Up to some point, it will be rational for the donkey to steer the float towards the larger bale. If it waits too long, however, its best choice may be to go to the smaller one, in order to avoid the high costs of rowing against the stream. By shifting the relative importance of interim and long-term costs and benefits, inadvertently crossing the threshold may doom a rational decision-maker to a lower long-term utility level. Late action can mean foregoing any rational motive to attempt reaching a superior long-run equilibrium, thus condemning the decider (for instance, society) to a gloomy future.

Thus motivated, this paper is devoted to a synthetic presentation of the properties necessary for the existence of multiple stable long-run stationary solutions and history dependence in optimal control problems where all potential externalities have been properly internalized, that is, in efficient optimal control problems. Particular attention is given to conditions under which the thresholds separating the stable long-run solutions take the particularly challenging form of so-called Skiba points, and to the properties of the thresholds. Section 2 introduces the class of dynamic optimization problems considered. Section 3 presents and discusses different necessary conditions for history dependence and the associated properties of the thresholds. Section 4 is devoted to the presentation of numerical methods for finding Skiba points and characterizing the global dynamics about these points. A numerical example is given. Section 5 gives a short survey of efficient dynamic economic models with multiple steady-states and history dependent outcomes. Section 6 concludes the paper. Most of the analysis is restricted to one-dimensional control problems. However, the paper also addresses the occurrence of multiple steady-states and thresholds when the state-space is two-dimensional. In that case, the thresholds are one-dimensional curves that generalize in a straightforward but non-trivial way the zero-dimensional (point) thresholds found in one-dimensional models.

At this place, let us make clear that we are not concerned with another important but unrelated problem in economic dynamics, namely, the problem of indeterminacy, see e.g. Majumdar et al. (2000). In the cases we are considering, and contrary to what characterizes indeterminacy, the optimal solutions are uniquely well-defined at almost every point. There exists, however, a set of points where the unique decision-maker may be indifferent between two or several of these solutions.

2 Framework and optimality conditions

In this paper, we consider inter-temporal optimization problems $P(x_0)$ of the type:

$$V(x_0) \equiv \sup_{u(t)\in U} \int_0^\infty e^{-rt} F(x(t), u(t))dt, \tag{1}$$

$$s.t.\ \dot{x} = f(x(t), u(t)), \quad x(0) = x_0, \tag{2}$$

where x is the state, u the control, U a compact set of admissible controls, t the time index, and $r > 0$ a discount rate. Furthermore, $F(x(t), u(t))$ is a return function, $f(x(t), u(t))$ describes the state dynamics, and $V(x_0)$ is the value function, i.e. the maximum aggregate present value of benefits for starting at $x(0) = x_0$. The problem is parameterized in terms of the initial conditions x_0, that are a crucial ingredient of a history dependent outcome.

In line with most of the relevant literature we restrict ourselves unless otherwise mentioned to one-dimensional models, i.e., $x \in \mathbb{R}$, and assume a scalar control, $u \in \mathbb{R}$. However, we will address extensions to the case $x \in \mathbb{R}^2$. To simplify the notation, we omit arguments whenever possible without risk of confusion. In particular, the variables are not indexed with time t in the rest of the paper. Similarly, the word optimal will be typically omitted. Thus, for example, an optimal trajectory (optimal solution) will be termed trajectory (solution).

Much of the presentation will be based on Pontryagin's *maximum principle*, and thus, carried out in terms of the current value Hamiltonian $H(u, x, \lambda)$:

$$H(u, x, \lambda) \equiv F(u, x) + \lambda f(u, x), \tag{3}$$

where λ is an co-state variable that can be interpreted as the dynamic shadow price of a marginal modification of the current state x. In terms of this Hamiltonian, the first-order conditions for an optimal policy u are given by the canonical system:

$$\begin{cases} \dot{x} = H_\lambda, \\ \dot{\lambda} = r\lambda - H_x, \end{cases} \tag{4}$$

together with the transversality condition:

$$\lim_{t\to\infty} e^{-rt}\lambda = 0, \tag{5}$$

and, assuming an interior solution, the Hamiltonian maximization condition:

$$H_u = 0. \tag{6}$$

Under standard assumptions, the system (4)-(6) associates to the current value of the state x an unique value of the co-state variable, $\lambda = \lambda(x)$. The optimal control u can likewise be expressed as a function of the current state x in terms of a *policy function* $u = u(x)$. We assume, unless otherwise specified, that $H_{uu} < 0$. This insures that the equation $H_u = 0$ can be solved for the control.

We speak of a *concave* model if H is jointly concave in the state x and the control u, i.e., if $H_{uu}H_{xx} - H_{ux}^2 \geq 0$. Otherwise, we speak of a *non-concave* model if $H_{uu}H_{xx} - H_{ux}^2 \leq 0$, or of a *convex* model if $H_{xx} > 0$. One should stress that we do not require concavity or convexity globally, but only over some compact set of interest for the concrete problem studied. Most importantly, note that joint concavity in x and u of the Hamiltonian is a globally sufficient condition for optimality. On the other hand, non-concave or convex models may violate, locally or globally, these sufficient conditions.

Besides the approach described above, another method based on the *Hamilton-Jacobi-Bellman* (HJB) *equation* is used at places, in particular, in Section 4. This approach relies on the fact that the value function $V(x)$ (which is assumed in this paper to be continuous) must satisfy the HJB functional equation:

$$rV(x) = \sup_{u \in U} \left[F(x,u) + V_x(x) f(x,u) \right], \tag{7}$$

where $V_x(x)$ is the derivative of $V(x)$ w.r.t. x. A third approach, discrete time dynamic programming, can be employed to study global dynamics and to detect thresholds based on a discretization of (7). Further details on these last two methods are given in Section 4.

Typically, the optimal long-run solutions of the control problem $P(x_0)$ – that is, the attractors of the optimally controlled system – will be saddle-points in the canonical space (x, λ).[3] Other types of attractors are possible, such as the origin if the feasible state-space is bounded by non-negativity conditions or, formally, $+\infty$ if the optimal trajectories grow without bounds. With some abuse of language, we will reserve the term *stable steady-state* to designate the value of the state variable x at any optimal attractor, independently of the specific form it may take.

In the context of a one-dimensional optimal control problem $P(x_0)$, a *threshold* is a value of the state variable x at which the decision-maker is indifferent between converging toward the one or the other of several stable steady-states. If it exists, the threshold will therefore lie between at least two stable steady-states x_L and x_R. Between the two stable steady-states, there will also always be an unstable stationary solution of the canonical system, whose x-value will be termed an *unstable steady-state* and designated by x_M.

Intuitively, it may be tempting to hypothesize that thresholds and unstable steady-states coincide. After all, the dynamics defined by (4)-(6) imply that the system will converge towards the one or the other stable steady-state depending upon whether the current state x is on the left or on the right of the unstable steady-state x_M. This intuition is correct if x_M is optimal in the sense that the pair $(x_M, \lambda(x_M))$ defined by (4)-(6) is optimal for the problem $P(x_0)$. However, it forgoes the fact that, as already mentioned, (4)-(6) provide only first-order

[3] Most non-economists (and some economists as well) may be puzzled by our naming a saddle-point an attractor. It is in the limited sense that, in the state-control space (a) the optimal steady-state is usually a saddle-point; and (b) the optimal trajectory leading to the optimal steady-state coincides with the stable branch of this saddle-point.

conditions for a possible optimum. Thus, the unstable steady-state can be non-optimal, in the sense that the pair $(x_M, \lambda(x_M))$ is non-optimal. If it is the case, the dynamics implied by (4)-(6) will typically not be optimal in a vicinity of the unstable steady-state, and threshold and unstable steady-state will normally differ. Note that, by definition, an inner stable steady-state is always optimal.

Thresholds that arise in connection with non-optimal unstable steady-states are usually called *Skiba thresholds* (or points, or sets), in honor of the pioneering work of Skiba (1978). Alternatively, they have also been named DNS thresholds, adding to Skiba's initial those of Dechert and Nishimura, who gave in their 1983 article the first exact proof of the existence of a Skiba point separating two basins of attraction in a one-state dynamic optimization model. Skiba thresholds are much more difficult to locate than those thresholds that are associated with optimal unstable steady-states, because there is no appropriate "local" equation to define them. In particular, the first-order conditions (4)-(6) do not provide the information needed to locate them exactly. It is therefore important to determine general conditions under which the thresholds are Skiba points and to find ways for finding their precise location.

The properties of the optimal control at a threshold are closely associated with the kind of threshold considered – Skiba point or not. In the case of a Skiba point, the control typically jumps at the threshold. Otherwise, it is everywhere continuous. Similarly, in the vicinity of a Skiba point, there may be several *candidate value functions* $\bar{V}(x)$, defined as the maximum aggregate present value of benefits when starting in x and converging towards a steady-state along a path satisfying the first-order conditions for a maximum of (1)-(2). We will see at a later place that both properties – discontinuity of the optimal control and multiple candidate value functions – are closely related. Finally, in the Skiba case, the threshold can be associated with either a node or a spiral source of the canonical trajectories. Otherwise, only nodes are possible.

These properties of the unstable steady-state x_M and associated threshold, on which we shall further elaborate in the next section, are summarized in Table 1. Remember that x_M is always optimal in concave models.

	$(x_M, \lambda(x_M))$ OPTIMAL	$(x_M, \lambda(x_M))$ NON-OPTIMAL
LOCAL CANONICAL DYNAMICS AT x_M	node	node or spiral
THRESHOLD	coincides with x_M	Skiba point; typically $\neq x_M$
OPTIMAL CONTROL	no jump at the threshold	typically jumps at the threshold
CANDIDATE VALUE FUNCTIONS $\bar{V}$	coincide with the true V	typically, several $\bar{V} \neq V$

Table 1: Main properties of the unstable steady-state x_M and of the threshold

It is important to note that the optimal solution does not satisfy the first-order conditions (4)-(6) not only at the Skiba point, but also in an (unknown) neighborhood of this point. The exact size of the neighborhood can only be determined numerically. For a global analysis of the truly optimal trajectories about the Skiba point see Wagener (2003).

3 Thresholds and necessary conditions

In this section, we classify and compare the necessary conditions for different types of thresholds separating optimal trajectories towards different stationary solutions. For the considered class of problems $P(x_0)$, convexity or at least non-concavity is usually considered to be the very property that causes the long-term behavior to depend on the initial state, i.e., that leads to history dependence. This history dependence due to 'increasing returns', 'positive feedbacks', etc., plays a central role in many policy related discussions, ranging among others from the choice of a technology to differences in economic development. The first subsection reviews this traditional road to multiple steady-states. The next subsection draws attention to the fact that this is not the only route and that a strictly concave framework does not rule out history dependent outcomes – a result that may appear surprising and, in any case, has been largely overlooked if not negated in influential economic literature. A separate subsection is devoted to the comparison of these two different generating mechanisms for multiple steady-states – the common denominator between the two being that the existence of an unstable steady-state is a necessary condition for history dependence. However, before comparing the two mechanisms, a subsection extends the analysis to the case of linear control models. The last subsection cursorily addresses extensions to higher dimensional systems.

3.1 Convexity and non-concavity

The theoretical contributions of Skiba (1978) and Dechert and Nishimura (1983) have sharpened our understanding of history dependent evolutions due to convexities and non-concavities. Our presentation of their results is based on the well known Ramsey model. In this model, u is consumption, F the utility from consumption, x the capital stock, and δ is the capital depreciation rate. The problem $P(x_0)$ takes the form:

$$\sup_u \int_0^\infty e^{-rt} F(u) dt, \tag{8}$$

$$s.t.\ \dot{x} = f(x,u) = \Phi(x) - u - \delta x,\ x \geq 0,\ x(0) = x_0, \tag{9}$$

where Φ is the production function, with $\Phi(0) = 0$ and $\Phi_x > 0$. The first-order conditions (4)-(6) yield the famous Ramsey rule:

$$\Phi_x = r + \delta. \tag{10}$$

In the standard case of global diminishing returns, i.e. when $\Phi_{xx} < 0$ for all x, the Ramsey rule defines for Φ satisfying the Inada conditions a unique non-trivial (i.e., $\neq 0$) stable steady-state since Φ_x is monotonically decreasing and ranges over all positive real numbers.

Let now introduce a non-concavity by assuming that the production function is locally convex, $\Phi_{xx} > 0$ for $x < \bar{x}$, $\Phi_{xx} < 0$ for $x > \bar{x}$, some $\bar{x} > 0$. That is, let assume increasing returns to scale for small capital stocks, $x < \bar{x}$. In that case, the Ramsey rule allows for two steady-states x_M and x_R, with $x_M < x_R$. The lower steady-state x_M lies in the convex domain of Φ, $x_M < \bar{x}$. The upper steady-state x_R lies in the concave domain, $x_R > \bar{x}$. Of these two steady-states, the one in the convex domain, x_M, is unstable. Although the pair $(x_M, \lambda(x_M))$ satisfies the first-order conditions (4)-(6), it is not optimal. Thus, there exists a Skiba threshold x^S that separates two domains of attraction towards the non-trivial steady-state x_R respectively towards the trivial steady-state $x_L = 0$. That is, $x(t) \to x_R$ for $x_0 > x^S$, and (asymptotically for F satisfying the Inada conditions) $x(t) \to 0$ for $x_0 < x^S$. Typically, the Skiba threshold x^S lies in a neighborhood of the unstable steady-state x_M, but does not coincide with it.

Locally increasing returns to scale underlie many economic models and are arguably the most typical cause for convexity or non-concavity, see Section 5. The economic sources of these increasing returns may among others arise from fixed costs in public infrastructure and networks (telecommunications, e.g.). Multiple steady-states are then possible if 'average' costs are higher for small than for large values of the stock. In the case of a telephone network, for example, the stock is the number of users. If this number is small, the average fixed costs per users will be typically much higher than if it were large.

3.2 Thresholds in concave models

The existing literature strongly suggests that the kind of convexities (or, at least, non-concavities) presented in the last sub-section are necessary in order to obtain multiple steady-states. To give a prominent example, Arthur (1989b) states that increasing returns give rise to lock-ins and thus to history dependence, but that the outcome is independent of history if technologies are subject to constant or diminishing returns. The purpose of this subsection is to correct this overall perception by showing that multiple steady-states and history dependence are possible in strictly concave inter-temporal optimization problems. Two early examples can be found in Kurz (1968) and Liviatan and Samuelson (1969). Nonetheless, this point has been neglected in important parts of the subsequent literature.

The following exposition draws on Feichtinger and Wirl (2000), who derive the necessary condition:

$$\frac{d\dot{x}}{dx} = f_x + f_u u_x > 0 \text{ at a steady-state} \tag{11}$$

for the existence of an unstable steady-state and of a threshold within a concave framework. This condition can only be satisfied if at a steady-state either:

$$r > f_x > 0 \tag{12}$$

or:

$$f_u u_x = f_u \frac{-H_{ux}}{H_{uu}} > 0. \tag{13}$$

Condition (12) requires 'growth', i.e., $f_x > 0$, but below the rate of discount. Thus, the standard Ramsey and the standard renewable resource models do not allow for an unstable steady-state, see (10).

Condition (13) also requires growth, but this growth is now indirectly induced by the optimal control. This requires that the mixed derivative characterizing the state-control interactions be the proper magnitude and sign: $H_{ux} > 0$ for $f_u > 0$, otherwise $H_{ux} > 0$. In this regard, note that Kurz (1968) emphasizes the 'growth' condition (12). Specifically, he introduces a wealth effect into the standard Ramsey model of optimal growth, that leads to an unstable steady-state with $r > f_x > 0$. On the other hand, while starting like Kurz with the Ramsey framework, Liviatan and Samuelson (1969) argue that an externality is not needed and rely on the control-state interactions (13) to insure the existence of an unstable steady-state.

To demonstrate the usefulness and the simplicity of the Feichtinger-Wirl approach, and in particular of the growth condition (12), consider the traditional Ramsey model with a strictly concave production function Φ. Assume, however, that the utility function includes wealth effects as in Kurz (1968) and Wirl (1994). In this formulation, the total utility F is the sum of the utility from consumption $v = v(u)$ and of a wealth effect $w = w(x)$, that is $F = v(u)+w(x)$. The Ramsey problem is then given by:

$$\sup_u \int_0^\infty e^{-rt} \left(v(u) + w(x)\right) dt, \tag{14}$$

$$s.t.\ \dot{x} = \Phi(x) - u - \delta x,\ \ x(0) = x_0. \tag{15}$$

The separable specification of F and f rules out (13) as a source for an unstable steady-state. Without the wealth effect, the Ramsey rule (10) must hold at an optimal steady-state, implying that the alternative condition (12) cannot be satisfied either. Indeed, the Ramsey rule requires $\Phi_x = r + \delta$, while (12) demands $\Phi_x < r + \delta$. The wealth effect, however, increases the stationary capital stock, thus decreasing Φ_x at the stationary solution. Thus, the model with wealth effect satisfies the growth condition $r > \frac{\partial \dot{x}}{\partial x} = \Phi_x - \delta > 0$ for any x between the traditional Ramsey rule, $\Phi_x = r+\delta$, and the maximum sustainable consumption, $\Phi_x = \delta$.

From the Hamiltonian:

$$H = v(u) + w(x) + \lambda \left[\Phi(x) - u - \delta x\right] \tag{16}$$

one derives the first-order conditions for interior solutions:

$$v_u - \lambda = 0, \tag{17}$$

$$\dot{\lambda} = (r + \delta - \Phi_x)\lambda - w_x. \tag{18}$$

Using for u in the state equation (15) the optimal control u^* determined by the maximum principle (17), $u^* = C(\lambda)$, $C_\lambda = 1/v_{uu} < 0$, yields the canonical equations in (x, λ) sketched in the phase diagram of Figure 1. The downwards sloping curve $\lambda = w_x/(r + \delta - \Phi_x)$ characterizes the $\{\dot{\lambda} = 0\}$ isocline. The $\{\dot{x} = 0\}$ isocline is U-shaped with its minimum at the point where stationary consumption is maximized ($\Phi_x = \delta$), and poles at the points of zero consumption ($x = 0$ and $\Phi(x) = \delta x$) for v satisfying the Inada conditions, suggesting the possibility of multiple steady-states. Straightforward numerical examples confirm that multiple steady-states can indeed arise – see e.g. Hof and Wirl (2000). Note that with the present model the unstable steady-state x_M is always optimal and coincides with the threshold.

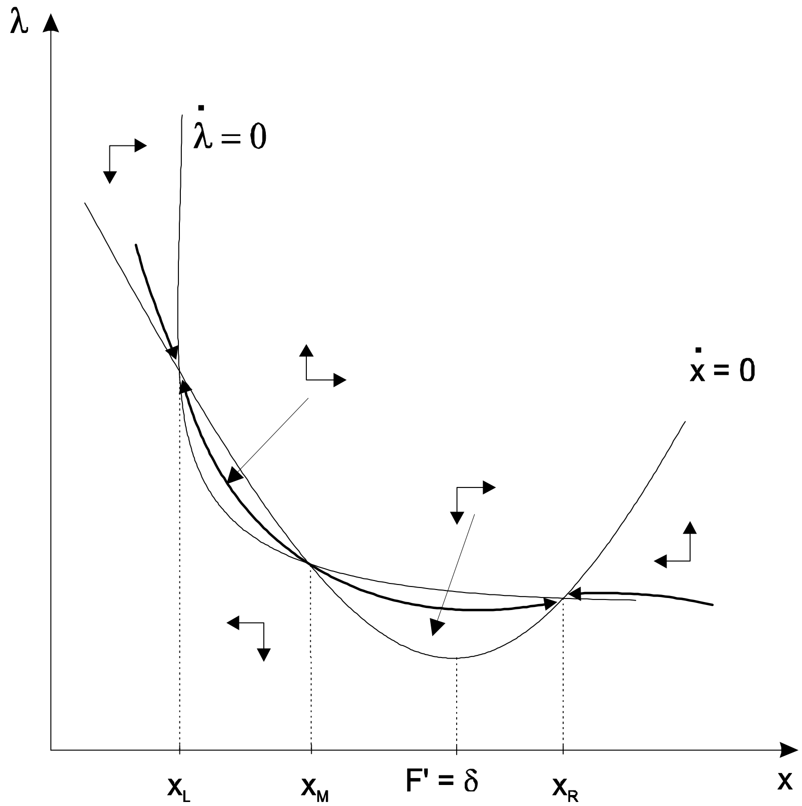

Figure 1: Phase diagram of the Ramsey model with wealth effects.

Some commentators have critically noted that conditions (11), (12) and (13) are only necessary for multiple steady-states. However, so are the familiar conditions of either convexity (with respect to the state) or lack of (joint) concavity. For example, the model of saving and growth with habit formation of Caroll et al. (2000) has a globally non-concave utility function[4]. Yet, the long-run

[4] This property is not explicitely pointed out by the authors.

outcome is unique.

3.3 Models linear in the control

A model that is linear in the control, $H_{uu} = 0$, has the same canonical equation system than in the previous case, with the only modification that the optimal control now depends discontinuously on the state and the co-states:

$$u = u(x, \lambda) = \begin{array}{l} \bar{u} \\ \text{singular arc} \\ \underline{u} \end{array} \quad \text{if} \quad H_u = F_u + \lambda f_u \begin{array}{c} > \\ = \\ < \end{array} 0,$$

where $\underline{u}$ and $\bar{u}$ denote lower and upper bounds on the control, $u \in U = [\underline{u}, \bar{u}]$. A steady-state is determined by the intersection of the singular arc, $H_u = 0$, with the $\dot{\lambda} = 0$ isocline. The singular arc is defined by $\lambda^{\text{sing}} = -F_u/f_u$. Because of $H_{uu} = 0$, a concave Hamiltonian implies $H_{ux} = 0$. Thus, λ^{sing} is a constant and the $\dot{\lambda} = 0$ isocline is monotonically declining. Consequently, multiple steady-states are impossible in a concave model, see Feichtinger and Wirl (2000).

Yet, if there is a local convexity with respect to the state, multiple steady-states are possible even in a separable model. Brock (1983) provides a nice example in the context of lobbying and entry deterrence. Moreover, even the milder condition of a lack of joint concavity allows for multiple steady-states and history dependence. Although concavity with respect to the state implies that the $\dot{\lambda} = 0$ isocline is monotonic, a singular arc depending on x may be sufficient for the isoclines to intersect more than once. This can be the case even if this dependence is linear, thus preserving concavity in x but not joint concavity in x and u. To recognize it, consider the following simple example of renewable resource extraction with an interaction in utility between the catch u and the biomass x:

$$F = \alpha u + \beta x + \gamma u x \text{ and } f = g(x) - u, \; g(x) = x(1 - x), \tag{19}$$

where $g(x)$ is the growth function for the biomass. The singular arc specifies the co-state as a linear function of the state, $\lambda^{\text{sing}} = \alpha + \gamma x$. The differential equation for the co-state is given by:

$$\dot{\lambda} = (r - g_x)\lambda - \gamma u - \beta.$$

Substituting the control that ensures a steady-state, $u = x(1 - x)$, into the differential equation for the co-state and solving for λ leads to:

$$\dot{\lambda} \mid_{\dot{x}=0} = 0 \Longleftrightarrow \lambda = \frac{\beta + \alpha g(x)}{r - g_x}.$$

A steady-state is determined by the intersection of $\dot{\lambda} \mid_{\dot{x}=0} = 0$ with λ^{sing}. For a proper choice of the parameters, there can be two positive steady-states, see Figure 2.

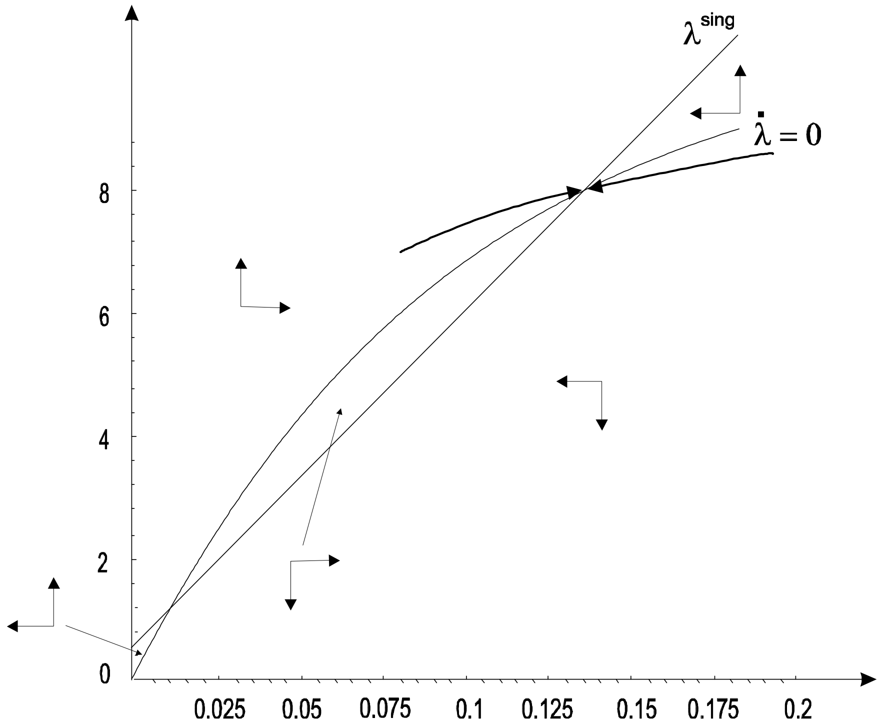

Figure2: Phase diagram for (19)
at the parameter values $r = 1.5, \alpha = 0.5, \beta = 0.05, \gamma = 50.$

3.4 Common features and differences

The economic implications of multiple steady-states, as we just discussed them, appear largely independent of the question whether or not the underlying model is concave. In any case, if the canonical system exhibits multiple steady-states, there exists a threshold, that is, a (set of) critical values of the state x with the following property: The optimal long-run outcome is different depending on which side of the threshold the current state lies. In the case of a one-dimensional system, it will be optimal to let x grow if its current value lies on the one side, to let it decline if it lies on the other side of the threshold.

Nonetheless, there are important differences between the unstable steady-states that arise in the concave and non-concave cases – recall Table 1. In order to recognize them, let us first investigate the eigenvalues of the Jacobian of the canonical equations. For the scalar state, scalar control optimal control model considered here, the eigenvalues of this Jacobian are invariably real for a concave

Hamiltonian, see Feichtinger and Wirl (2000). Thus, in the concave case, any unstable steady-state is a node. In the non-concave or convex cases, both a node and a spiral are possible. That is, the type of local dynamics around the unstable steady-states, node or spiral, does not allow for properly differentiating between the concave and the non concave/convex cases. Nonetheless, much of the economic literature on non-concave applications neglects the possibility of nodes and assume, without verifying whether or not it is the case in the particular model considered, that the unstable steady-states are spirals of the canonical system.

Another error commonly encountered is that history dependence necessarily involves a jump in the control at the threshold. This is not generally true. A jump is impossible in the concave case, since the optimal control is unique. Hence, a jump can occur only in non-concave and convex models.

Finally, the unstable steady-states are always optimal and invariably coincide with the threshold in the concave case. In the non-concave and convex cases, this is usually not true. The unstable steady-states are not optimal and do not coincide with the thresholds, which are in that case Skiba points.

These last two points, the (non) existence of a jump and the coincidence or non-coincidence of the threshold point with an unstable steady-state, can be best illustrated in terms of the candidate value functions $\bar{V}(x)$ defined in Section 2. Suppose there are three steady-states, $x_L < x_M < x_R$, with x_M unstable.

In the case of a concave framework, any candidate value function $\bar{V}(x)$ is globally optimal and unique – that is, it is the unique true value function $V(x)$ for the problem of interest, see the upper drawing in Figure 3. Therefore, there is a unique control that satisfies the first-order conditions at the unstable steady-state x_M. This control and $(x_M, \lambda(x_M))$ are optimal.

In a non-concave or convex framework, at least two candidates for the value function, say $\bar{V}^L$ and $\bar{V}^R$, exist, the first being associated with x_L and the second with x_R. Since the problem of interest is a maximization, one should choose for any given initial state x_0 the solution that yields the highest possible payoff so that the (by definition unique) value function is given by $V(x) = \sup\left\{\bar{V}^L(x), \bar{V}^R(x)\right\}$. Therefore, if $\bar{V}^L(x_0) < \bar{V}^R(x_0)$, then it is optimal to choose the optimal control path that leads ultimately to x_R, and if $\bar{V}^L(x_0) > \bar{V}^R(x_0)$ the one leading to x_L. At the value x^S where $\bar{V}^L$ and $\bar{V}^R$ intersect, i.e. for which $\bar{V}^L(x^S) = \bar{V}^R(x^S)$, one is indifferent between heading towards x_L or towards x_R. If indeed several candidate value functions do exist, the threshold value x^S will only incidentally coincide with the unstable steady-state x_M. Moreover, $(x_M, \lambda(x_M))$ will not be optimal. That is, the threshold in this case is a Skiba point x^S. This can be seen by substituting the stationary control satisfying the first-order conditions associated with x_M into the steady-state equation. The payoff at this steady-state falls short of the maximum, see the lower drawing in Figure 3. Since the value functions cross at x^S, the derivatives of the value functions typically differ. But, according to the Hamilton-Jacobi-Bellman equation, the optimal control depends on this derivative. Thus, it jumps at x^S.

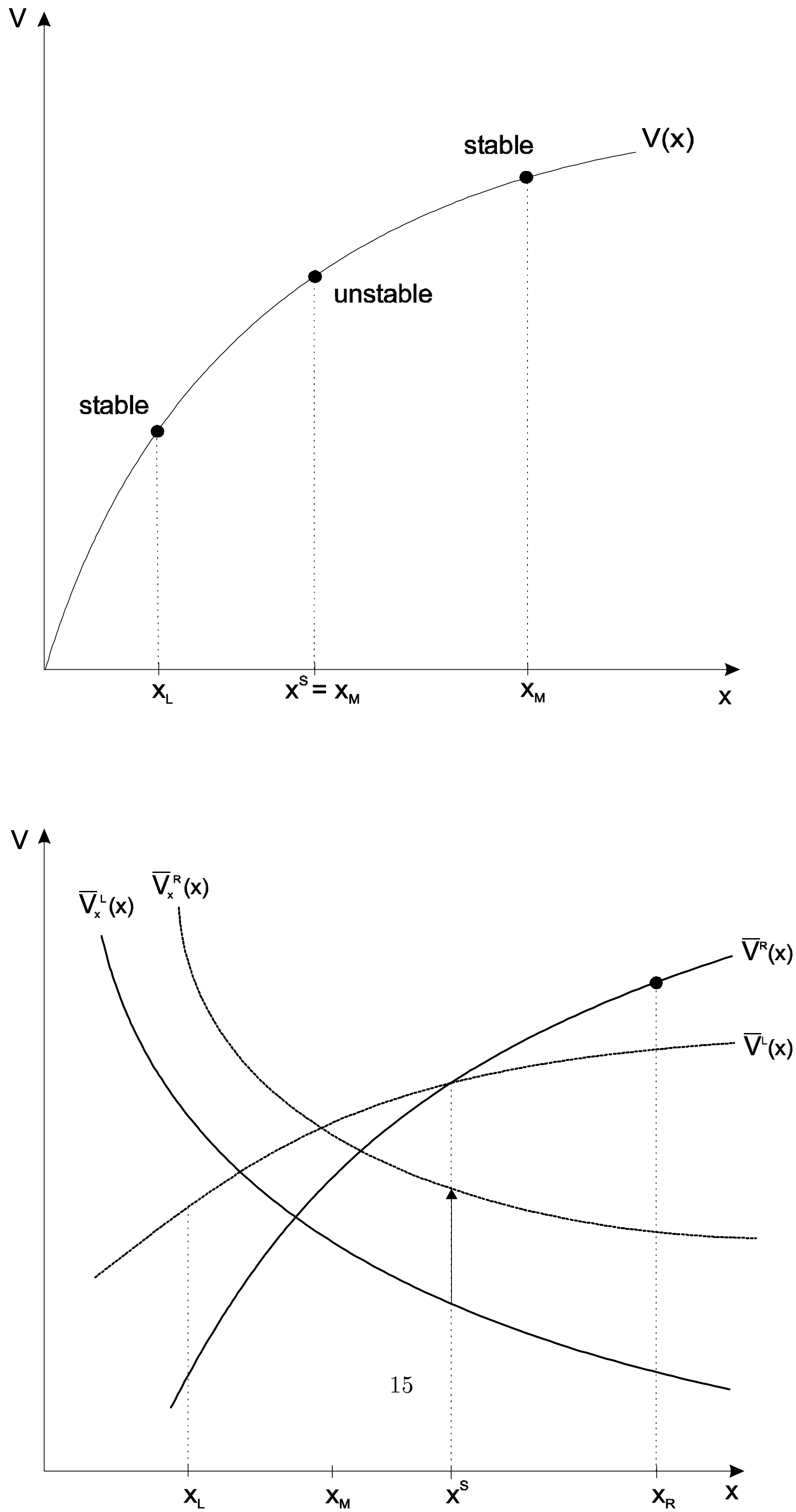

Figure 3: Comparing unique (concave) and multiple candidate value functions.

As mentioned earlier, concavity insures that there is a unique continuous candidate value function and thus, that the policy function is continuous. But what can be said about the continuity of the policy function in the non-concave or convex cases? Matters are clear when the unstable steady-state is a spiral. A simple inspection of the phase diagram shows that in that case the steady-states cannot be connected by a continuous policy function. Interestingly enough, this is not necessarily true when the unstable steady-state is a node. Consider the relative adjustment cost framework in Feichtinger et al. (2001):

$$\sup_u \int_0^\infty e^{-rt}[v(x) - C(\frac{u}{x})]dt,$$

$$\dot{x} = u - \delta x, \; x(0) = x_0,$$

where v is the concave gross profit function, x the capital stock, u the gross investment, C a convex cost function with the ratio of replaced capital as argument. This framework seems particularly well suited to trace out the rather subtle points we are addressing here, because (a) it insures the existence of multiple steady-states (for the quadratic specification $v = x - \frac{1}{2}x^2$ and $C = \frac{1}{2}\gamma\left(\frac{u}{x}\right)^2$, for example, the model admits three steady-states) and (b) the unstable steady-state can fall into the concave or the non-concave domain and be either a node or a spiral. In the case of an unstable node in the non-concave domain, Hartl et al. (2003) present a numerical example with a phase diagram that allows for a continuous connection between the steady-states, see Figure 4. Thus, for specific values of the parameters, a unique candidate value function may indeed exist. This point clearly requires further research. In any event, it suggests the need for correcting the loose and often erroneous statements found in the literature that often assumes the existence of an unstable spiral without carrying out the necessary eigenvalues analysis.

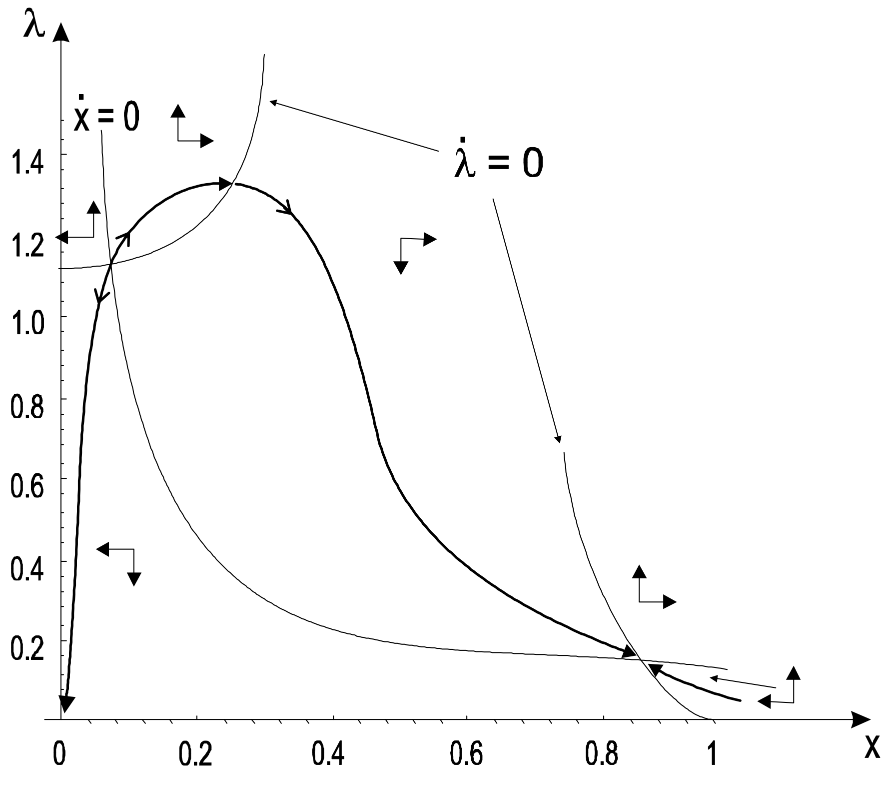

Figure 4: Phase diagram for the relative adjustment cost framework, quadratic specification, $\gamma = \frac{3}{2}, r = 1, \delta = 0.1.$

3.5 Higher order systems

We consider only such two-dimensional systems, $x \in \mathbb{R}^2$, that can be derived from one-dimensional ones using the embedding approach developed in Feichtinger et al. (1994). With this approach, the originally one-dimensional problem is transformed into a two-dimensional one by introducing control adjustments costs, denoted v. For simplicity's sake, these costs are assumed here to be quadratic:

$$\sup_{u(t)\in U} \int_0^\infty e^{-rt} \left(F(x,u) - \frac{1}{2} c v^2 \right) dt, \tag{20}$$

$$s.t.\ \dot{x} = f(x,u),\ x(0) = x_0,\ \dot{u} = v,\ u(0) = u_0, \tag{21}$$

Since $v = 0$ at a steady-state, the original one-dimensional and the derived two-dimensional problems both have the same steady-states.

An unstable steady-state of the original problem remains unstable for the derived problem, since $\det(J) > 0 \Longleftrightarrow \det(\widetilde{J}) < 0$, where J is the determinant of the canonical equations system of the original, and $\widetilde{J}$ the Jacobian of the four-dimensional canonical equations system of the derived problem. It is impossible to stabilize an unstable system by introducing adjustment costs.

The converse is not true. Adjustment costs may destabilize an otherwise stable steady-state, see Feichtinger et al. (1994). In particular, in the concave case, adjustment costs can transform a stable steady-state into an unstable steady-state or into a limit cycle (e.g. a Hopf cycle) if the growth condition $r > f_x > 0$ is satisfied. This may appear counter-intuitive, since at the original stable steady-state no control adjustments are necessary, while at the derived unstable one costly adjustments are permanently needed.

Thus, there are two possible causes for the existence of an unstable steady-state in the derived problem: (a) the original problem has one; or (b) the original problem has a stable steady-state that becomes unstable due to the adjustment costs. The growth condition (11) must be satisfied in both cases. Thresholds and history dependence arise only in the first case (a), that is, when the original steady-state is unstable, $\det(J) > 0$. As previously indicated, this is the case if and only if $\det(\widetilde{J}) < 0$. The second road to instability (a) does not lead to thresholds.

The condition for an unstable steady-state, $\det(\widetilde{J}) < 0$, is equivalent to one eigenvalue of the Jacobian being negative and the three others either being positive or having positive real parts, see Dockner (1985). Hence, an unstable steady-state remains conditionally stable along a one-dimensional manifold M of initial conditions. This is illustrated in Figure 5, that shows a situation with two (saddle-point) stable steady-states and an unstable one, indicated by dots. The unstable steady-state can be reached from any initial condition along the dotted line, that is, from any point on the manifold M. This extends by one dimension the well-known property that in the one-dimensional case the system remains at the threshold if it starts there. In a concave framework, unstable steady-states are optimal, so that the manifold M is also the threshold that separates the domains of attractions of the stable steady-states.

For a convex model, similarly to the one-dimensional case, the threshold is given by the intersection of the candidate value functions associated with the long-run outcomes. The projection of this intersection onto the state space, a one-dimensional manifold, is the Skiba threshold that separates the domains of attractions of the stable steady-states. This threshold can differ from the one-dimensional manifold M, if the unstable steady-state and the corresponding stationary control are not optimal.

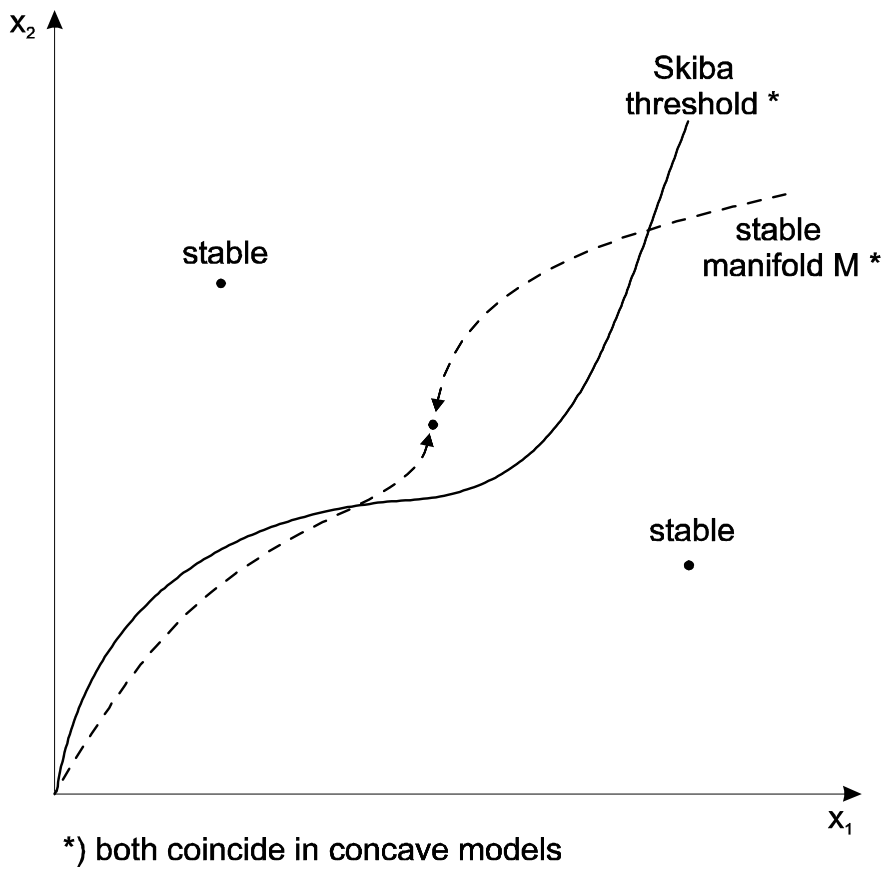

Figure 5: Thresholds in two-dimensional models.

Thus, the concept of a separating threshold between stable steady-states defined for one-dimensional models extends naturally, and directly to the case of two dimensions. The main difference is that it is no longer a point, but a curve, that separates the domains of attractions. Also, in the one-dimensional case, the optimal policy is necessarily unique on the right and on the left of a Skiba point. Two dimensional Skiba thresholds, by contrast, can separate several roads to the same equilibria. Finding the separating curve is fairly straightforward in the concave case, but not trivial, since it is the stable manifold associated with the unstable steady-state. If the model is not concave, its computation is even more involved, as shown in the next section.

4 Numerical Methods for Detecting Skiba Points

A rigorous study of a dynamic model with multiple steady-states requires locating the thresholds analytically. When these thresholds are Skiba points, there is no appropriate "local" equation to define them. In this case, the Skiba points have to be determined numerically. In this section, we present three methods that can be used to that purpose.

The **first method** uses the Hamilton-Jacobi-Bellman (HJB) equation to solve the basic control problem $P(x_0)$ numerically. We summarize the corresponding algorithm as applied by Semmler and Sieveking (1999). This algorithm, due to Brooks Ferebee, implies three steps:

1. The first step consists in finding all the steady-states x_i, $i = 1, 2, ...$ To do so, compute the optimal feedback control $u(x)$ that would keep x constant over time, and define $g(x)$ as $g(x) := F(x, u(x))$. To obtain the steady-states x_i, use the envelope condition for the continuous-time control problem $P(x_0)$:
$$r\frac{\partial f_o(x,u)}{\partial u} + g_x(x) = 0, \qquad (22)$$
where $g_x(x) := \partial g(x)/\partial x$, and solve for the x_is. Remember that in a non-concave problem not all $(x_i, \lambda(x_i))$s are necessarily optimal.

2. In a second step, use the first-order conditions for a maximum w.r.t. u of the right-hand-side of the stationary HJB-equation:
$$rV(x) = \sup\left[F(x,u) + V_x(x)f(x,u)\right] \qquad (23)$$
to obtain, if possible, an explicit expression for $V_x(x)$ as a function of x and $V(x)$:
$$V_x(x) = G(V(x), x), \qquad (24)$$
where $V_x(x)$ is the derivative of $V(x)$ with respect to x. For each steady-state x_i, solve the differential equation (24) using as boundary condition:
$$V(x_i) = \int_0^\infty e^{-rt} g(x_i)dt = \frac{1}{r}g(x_i), \qquad (25)$$
thus obtaining solutions $\bar{V}^i$ of (24).

3. All the solutions obtained under step 2 satisfy the stationary HJB-equation, that is, the necessary conditions for a value function. However, since they were computed starting from steady-states x_i that are not necessarily optimal, they do not automatically satisfy the sufficient conditions. In that sense, they are just (local) candidates for the true value function. The

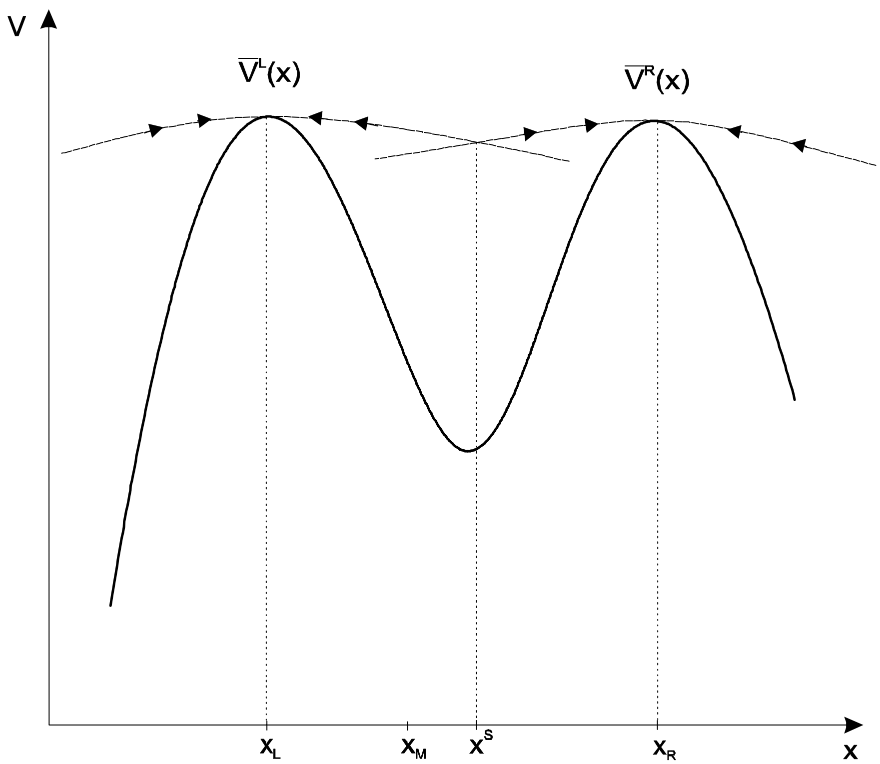

Figure 1: Figure 6: Skiba point and global dynamics computed through the HJB-equation.

sufficient conditions will be satisfied at any given x by a solution $\bar{V}^i$ iff $\bar{V}^i(x) \geq \bar{V}^j(x)$, all $j \neq i$. The true value function $V(x)$ is found in a third step by solving:

$$V(x) = \sup_i \bar{V}^i. \tag{26}$$

In other words, $V(x)$ is defined piecewise by taking the upper envelope of the different candidate value functions $\bar{V}^i$. The Skiba point(s) x^S is (are) located where the solution $\bar{V}^i$ that defines the upper envelope changes, that is, at the intersection of candidate solutions $\bar{V}^i$, see Fig. 6.

The main achievement of this algorithm is to find the location of the Skiba points from the solutions of the HJB-equation. The outer envelope defined by (26) determines the optimal global dynamics, that is, the history-dependent solutions. Note that knowing V permits to calculate the optimal control $u(x)$ in feedback form using the HJB-equation.

The **second method** is based on the maximum principle and the Hamiltonian. The necessary conditions provided by the Hamiltonian typically imply the dynamic properties shown in Figure 7.

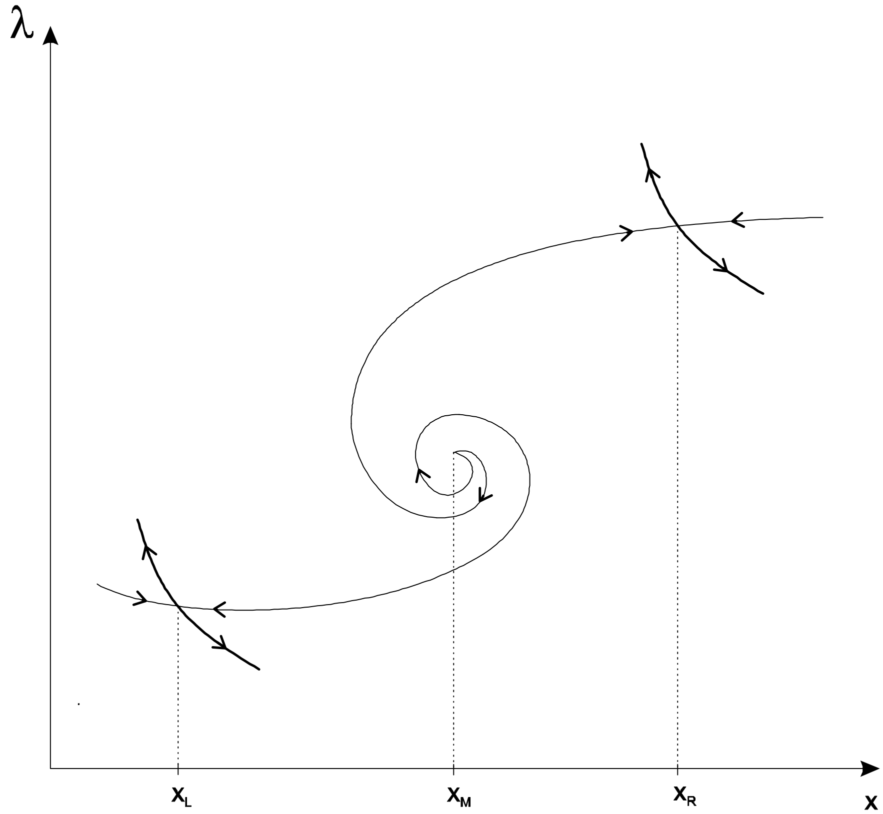

Figure 7: Local dynamics about the candidates.

As stressed throughout this paper, the steady-state x_L is usually a saddle-point, x_M is an unstable node or focus, and x_R is again a saddle-point. There are connecting orbits from x_M to x_L and x_R. Thus, in general, one can proceed as follows to obtain the global dynamics from the Hamiltonian $H(\cdot)$ associated with the problem $P(x_0)$.

1. Compute the steady-states x_i.
2. Compute the local canonical dynamics about the steady-states x_i.
3. Compute the integrals along the stable manifold from the right and from the left of the middle unstable steady-state. The intersection of the two integral curves is the Skiba point, as shown in Figure 8.

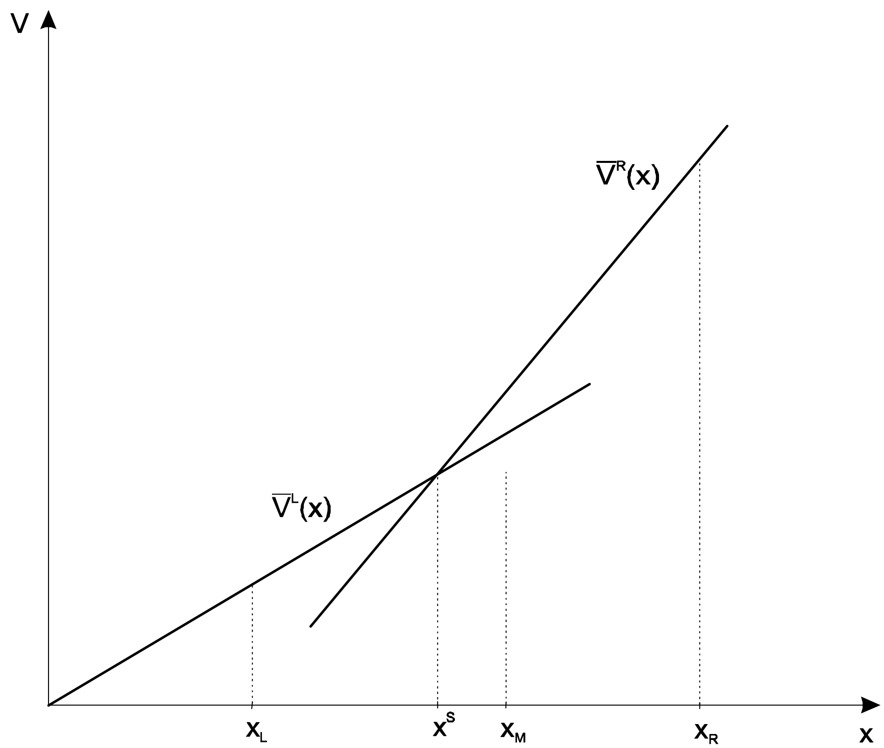

Figure 8: Skiba points and global dynamics computed through the Hamiltonian.

The point where the two integral curves intersect is the Skiba point x^S because at this point the payoffs for going to x_L are the same as for going to x_R. At the middle candidate x_M there are typically two solutions satisfying the first-order conditions for optimality. However, one is superior to the other. For details, see Beyn et al. (2001), that also indicate how to compute thresholds in higher dimensional systems.

This second method has been suggested by Skiba (1978), analytically further pursued by Brock and Malliaris (1989) and Brock and Starret (1999), and numerically implemented by Beyn et al. (2001), and by Haunschmied et al. (2003). Although it is useful for computing the global dynamics, it has shortcomings. The precision with which the Skiba point can be computed depends crucially on the approximation of the connecting orbits, that is, of the stable manifolds for the steady-states x_L and x_R as shown in Figure 8. In order to obtain the correct integrals, the connecting orbits should be precisely computed on grid points in the state space, starting from x_M and moving left to x_L and right to x_R. These shortcomings, however, do not weigh heavily in one-dimensional problems.

A **third method,** dynamic programming, can also be employed to compute the Skiba points. Using the continuous dynamic programming equation (7) is equivalent to iterating on the value function. If the iteration is properly done and converges, the value thus obtained will be greater at a non-optimal steady-state x_i than the value of V resulting from applying the optimal feedback control $u(x_i)$ that would keep $x = x_i$ constant. This allows to check the optimality of the candidate by direct inspection.

Generically, the dynamic programming method is more efficient at finding strong attractors (for example limit cycles) than at detecting thresholds or Skiba points, see Sieveking and Semmler (1997). Strong attractors are not influenced severely by numerical errors such as rounding inaccuracies. By contrast, such errors can considerably impair the search for a Skiba point, as this search amounts to numerically locate a point in the state space where the control u starts changing direction. The use of dynamic programming on a grid for the state and control equations generates numerical rounding errors that pile up in the iteration of the value function, and also impact the control u. To rely on dynamic programming to numerically find the Skiba points, it is necessary to have trustworthy estimates of the associated error bound. The problems of discretization and estimation of error bounds are discussed in Semmler and Sieveking (1999) and Grüne and Semmler (2002). This last paper demonstrates the usefulness of using an algorithm with flexible grid size about the region where the Skiba point is expected to lie in order to efficiently find thresholds through dynamic programming.

4.1 A numerical example

In this sub-section, we use the model with relative adjustment cost of section 3.4. to summarily demonstrate the usefulness of the HJB-equation in computing value functions and Skiba points. Details are given in Kato and Semmler (2001).

The present value problem to be solved is:

$$V(x) = \sup_u \int_0^\infty e^{-rt} F(x,u) dt \tag{27}$$

$$= \sup_u \int_0^\infty e^{-rt} \{v(x) - C(\frac{u}{x})\} dt, \tag{28}$$

$$\text{s.t.} \quad \dot{x} = u - \delta x, \;\; x(0) = x_0, \tag{29}$$

where $v = x - \frac{1}{2}x^2$ is a concave gross profit function, x the capital stock, u the gross investment, and $C(\frac{u}{x}) = \frac{1}{2}\gamma\left(\frac{u}{x}\right)^2$ a convex cost function with relative adjustment costs as argument. Thus:

$$F(x,u) = x - \frac{1}{2}x^2 - \frac{1}{2}\gamma\left(\frac{u}{x}\right)^2. \tag{30}$$

Remember that the model admits three steady-states, and that the unstable one can fall into the non-concave domain. This is the case for the parameter values used in the numerical computations to be presented below, that is, $\gamma = 20, \;\; \delta = 0.1$ and $r = 0.05$.

From (29) one recognizes immediately that at a steady-state $u = \delta x$. Substituting for u in (30) gives the function $g(x)$:

$$g(x) = x - \frac{1}{2}x^2 - \frac{1}{2}\delta^2\gamma. \tag{31}$$

The stationary HJB-equation is:

$$\begin{aligned} rV(x) &= \sup_u \left[F(x,u) + V_x(x)(u - \delta x)\right] & (32) \\ &= \sup_u \left[x - \frac{1}{2}x^2 - \frac{1}{2}\gamma\left(\frac{u}{x}\right)^2 + V_x(x)(u - \delta x)\right]. & (33) \end{aligned}$$

As previously stated, we can compute the value functions and thresholds in three steps.

Step 1: Identify the steady-states x_i. From the envelope condition:

$$r\frac{\partial F(x,u)}{\partial u} + g_x(x) = 0 \tag{34}$$

one obtains:

$$u = \frac{1-x}{r\gamma}x^2. \tag{35}$$

Using the stationarity condition $u = \delta x$, see (29), this last equation defines three steady-states:

$$x_i = \begin{cases} 0 \\ \frac{1 \pm \sqrt{1-4r\gamma\delta}}{2} \end{cases}, \tag{36}$$

that is, for the parameter values indicated above:

$$x_L = 0, \quad x_M = 0.112702, \quad x_R = 0.887298.$$

Step 2: Solve the stationary HJB equation (32) starting from above steady-states. From the first order conditions for a maximum w.r.t. u of the RHS of (32) one obtains:

$$V_x(x) = \gamma\frac{u}{x^2}. \tag{37}$$

Substituting (37) into (32) gives after some manipulations:

$$V_x(x)^2 - 2\gamma\frac{\delta}{x}V_x(x) + 2\frac{\gamma}{x} - \gamma - 2\gamma\frac{r}{x^2}V(x) = 0. \tag{38}$$

Solving (38) in terms of $V_x(x)$ yields the ordinary differential equation in x:

$$V_x(x) = \frac{\gamma\delta}{x} \pm \sqrt{(\frac{\gamma\delta}{x})^2 - (2\frac{\gamma}{x} - \gamma - 2\frac{\gamma r}{x^2}V(x))} \quad \text{for} \quad \begin{matrix} x \leq x_i; \\ x > x_i, \end{matrix} \tag{39}$$

that is, the function $G(V(x), x)$, see (24).

We solve this ODE using some iterative method starting at the three steady-states x_L, x_M, and x_R, taking as initial values:

$$V(x_i) = \frac{1}{r}[x_i - \frac{1}{2}x_i^2 - \frac{1}{2}\gamma\delta^2], \quad x_i = x_L, x_M, x_R. \tag{40}$$

Step 3: Find the global value function. The different solutions $\bar{V}^L, \bar{V}^M$ and $\bar{V}^R$ obtained by solving (39)-(40) are candidate value functions for the problem of interest. Notice that (39) implies that starting at any steady-state x_i there are two possible solutions for the ODE and thus two candidate value functions. The true value function is given by the upper envelope of these candidates:

$$V = \sup\left[\bar{V}^L, \bar{V}^M, \bar{V}^R\right]. \tag{41}$$

The candidate solutions $\bar{V}^L$ and $\bar{V}^R$ that define the upper envelope for the problem of interest are shown in the left graph of Figure 9.[5] Thus, the candidate value function $\bar{V}^L$ starting at $x_L = 0$ coincides with the true value function until it intersects the candidate $\bar{V}^R$ at the Skiba point $x^S \simeq 0.102$. The candidate $\bar{V}^R$ is the (prolongation of the) true value function from this point on. The Skiba point x^S lies in the vicinity of $x_M = 0.112702$, but does not coincide with it. There is a discontinuity in the optimal control at the Skiba point. This discontinuity is clearly seen on the right graph of Figure 9, that is a blow up of the left graph around x^S.

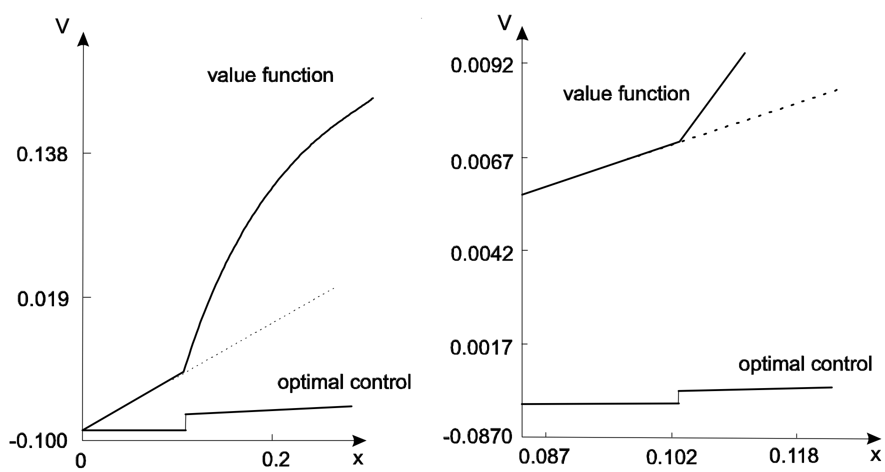

Figure 9: Value function and optimal control.

The figures confirm that the unstable steady-state $x_M = 0.112702$ is not optimal and do not coincide with x^S. The other steady-states $x_L = 0$ and

[5] The candidate value functions $\bar{V}^L$ and $\bar{V}^L$ and the Skiba point where computed using a dynamic programming algorithm with flexible grid size, see Grüne and Semmler (2002) and earlier remarks.

$x_R = 0.887298$, on the other hand, are optimal attractors. This last result, of course, can be trivially inferred from the fact that our simple example admits exactly three steady-states, namely, one unstable steady-state x_M between the two stable steady-states x_L and x_R.

5 Economic applications

The recognition that multiple steady-states and history dependence can readily arise in economic models is not new. However, for a long time, interest for such multiple equilibria has been fairly subdued. To put it crudely, multiple steady-states were politically incorrect in the 1960s and 1970s. Much effort was therefore done by authors such as Brock (1973, 1974 e.g.) to delineate sufficient conditions that would exclude them. Only in the 1980s did dynamic models with multiple steady-states and history dependence became fashionable, motivated by the contributions of Skiba (1978) and Dechert and Nishimura (1983) and, more importantly perhaps, by the influential but inexact assertion of Arthur (1989, 1994b) that increasing returns to scale lead to outcomes highly sensitive to initial conditions while decreasing returns ensure uniqueness.

Multiple equilibria and history dependence can occur in a wide variety of formal frameworks. In the following, we give a brief review of economic models that correspond formally to our basic control problem without externalities $P(x_0)$, and that exhibit multiple equilibria.

5.1 Intertemporal optimization under convex-concave technologies

In the case of efficient economies the assumption of local non-concavities is by far the most common cause for multiple steady-states. One of the simplest examples is given by the one-dimensional models of capital accumulation with a convex-concave production function that can be found, among others, in the theory of economic development. In this context, the hypothesis of a convex-concave production function is usually justified by the presence of social inputs such as institutions or human capital. This hypothesis is responsible for the existence of a Skiba point between a stable high and a stable low income steady-state. Thus, these models can explain the coexistence of countries with low and of countries with high per capita income as resulting solely from different initial conditions. A widely used version of a convex-concave production function can be found in Skiba (1978) and Azariadis and Drazen (1990).[6] Similarly, multiple steady-states can arise in a one-capital-good model if nonlinear adjustment costs of investment are assumed as in Blanchard (1983), see Semmler and Sieveking (1999). Multiple steady-states can also arise in a one-capital-good model when the adjustment cost is a nonlinear function of the change of investment as, for example, in Haunschmied et al. (2003). This last model is a two-state-variables

[6] Econometric tests of such models with threshold effects are undertaken in Durlauf and Johnson (1995) and Bernard and Durlauf (1995).

model, significantly complicating the determination of the threshold, which is now two-dimensional.

The nowadays popular idea that increasing returns to scale can be responsible for multiple equilibria and history dependence can be traced in numerous applications beyond the narrow core of economics. To name a few, Altmann (2000) applies this idea to sociology, Cowan and Gunby (1996) to pest control, and Krugman (1995) and Fujita et al. (1999) to economic geography.

5.2 Per capita variables

The use of ratios between primitive variables may lead to multiple equilibria. Thus, Schelling (1967, 1973) assumes that the probability that an agent engages into an illegal transaction increases with the total number of agents that engage in the transaction. This results in three equilibrium points, two stable boundary ones – corresponding respectively to a "clean" and to a "dirty" society – and an unstable interior one. In this context, Andvig (1991) introduces what he calls a Schelling diagram to explain observed variations in corruption frequency, and gives several other examples of multiple equilibria in the context of the economics of crime. While these considerations are essentially static, Feichtinger et al. (2002) obtain a similar structure of long-run equilibria in an intertemporal model of law enforcement based on the model of Caulkins (1993). The level of enforcement depends on the law enforcement effort relative to the size of the population of offenders, that is, on the per capita enforcement effort – a mechanism that has been named enforcement swamping in the context of illicit drug consumption – see Kleiman (1993), Caulkins et al. (2000), Tragler et al. (2001). This dependency implies multiple steady-states, unstable nodes or focuses, and Skiba thresholds. The result is fairly general. The use in the objective function of "per capita" quantities, generically defined as a control variable divided by a state variable, usually generates a convexity of the Hamiltonian with respect to the state and may easily lead to multiple equilibria. See Borisov et al. (2000), Kort et al. (1998), and Feichtinger and Tragler (2002) for related works in the context of law enforcement. Another interesting field of application of the "per capita" threshold generating mechanism are relative adjustment costs, see the framework in Feichtinger et al. (2001) and Hartl and al. (2003). As mentioned in Section 3.4, the thresholds in this model can fall in the concave as well in the convex domain. The model thus provides a link between the different threshold generating mechanisms. Formally related is Gould's (1970) second diffusion model, an early forerunner of a model exhibiting a Skiba point.

5.3 Regulatory economics

Dynamic convex-concave models with history dependent outcomes can also be found in regulatory economics. Brock and Dechert (1985) investigate dynamic Ramsey pricing, with the interesting result that the maintenance of a public service – say, of a railroad network – depends on the initial conditions (passenger trains almost disappeared in Australia and the United States but are omnipresent in Europe). Furthermore, the viability of a service does not imply that a private firm will maintain the service. Profit maximization may imply liquidation in the long-run although a viable and stable steady-state exists. These properties stem from convex-concave functions, with the convexity resulting from locally increasing returns. Along similar lines, Dechert (1984) considers the familiar Averch-Johnson effect. Brock (1983) investigates a positive problem, lobbying, within a convex-concave setting. Their models assume (locally) increasing returns to scale and thus support the familiar results of multiple steady-states and Skiba points.

5.4 Models of addiction

In the context of addiction, the possible occurrence of thresholds ('cold turkeys') separating multiple steady-states was already noted by Becker and Murphy (1988). However, the linear-quadratic framework used by these authors precluded their materialization. Not so in the more recent models of Orphanides and Zervos (1998), who use a generalized non-linear framework with convex-concave preferences. Here, three steady-states exist, the middle one being unstable. At the lower steady-state, there is no drug consumption. At the higher one, there is addiction. Due to small shocks in drug consumption or to the impact of enforcement policies, the system can converge towards the one or the other of these last two steady-states. The threshold separating the optimal trajectories leading to either the lower or the higher stable steady-state does not necessarily coincide with the middle unstable one. For a recent related contribution, see Gavrila et al. (2003).

5.5 Monetary policy models

Some recent monetary policy models exhibit similar properties, such as Benhabib et al. (1998). In this model, consumers' welfare is affected positively by consumption and cash balances and negatively by the labor effort and an inflation gap from some target rates. The model admits an unstable steady-state surrounded by two stable ones with high respectively low inflation rate. Moreover, there can be indeterminacy in the sense that any initial condition in the neighborhood of one of the unstable steady-states is associated to an optimal path. The same kind of dynamics is found in Greiner and Semmler (1999).

5.6 Renewable resources

Dynamic models of renewable resources with two state variables can easily exhibit multiple steady-states. In the two-resources model of Sieveking and Semmler (1997), the resource dynamics may have at least three steady-states depending on the type of interaction between the resources – competitive, predator-prey or cooperative. Here again, the middle steady-state is unstable, while the outer two are saddle-points. Similarly, multiple steady-states have been shown to exist in ecological management problems, see Lewis and Schmalensee (1982), Tahvonen and Salo (1996), Tahvonen and Withagen (1996), and Rondeau (2001). Recently, Brock and Starret (1999), Dechert and Brock (1999), [7] uses bifurcation techniques to investigate the occurrence of thresholds in this "shallow lake model". One of the interesting results of his very thorough investigation is that an unstable steady-state can exist although there is no threshold – a possibility that was previously totally ignored in the literature. While all above papers rely, in the tradition of Skiba (1978), on a convex-concave maximization problem to generate thresholds a few, e.g. Wirl (2003), use a strict concave framework to obtain thresholds that others, such as Ayong Le Kama (2001), overlook.

5.7 Related models

The emergence of multiple equilibria in dynamic optimization models is even more frequent if one relaxes the previously hypothesis that all potential externalities are properly internalized, if there are anticipations, and/or if one allows for strategic interactions among multiple decision-makers – that is, if one considers differential games rather than optimal control problems.

The externality route is, for example, followed in the endogenous growth literature initiated by Lucas (1988) and Romer (1990) – see Chamley (1993), Benhabib and Perli (1994), Xie (1994) and Ladron-de-Guevara et al. (1999), Santos (1999), and many others. In these models, the multiple steady-states provide an explanation for the possibility of different growth paths and the occurrence of "poverty traps". See Azariadis (1986), Marrewijk and Verbeek (1983).

If there is an externality, rational agents will forecast its evolution. Depending upon the initial conditions, this can give rise to different expectations and long-run outcomes. This point was first made in Krugman (1991). A recent application to CO2 permits can be found in Liski (2001). Another related example is Arthur's (1994a) well-known El-Farrol problem.

In differential games, multiple feedback Nash equilibrium steady-states can arise even in the case of a linear-quadratic game with one state variable. This results from the fact that the differential equation implied by the Hamilton-Jacobi-Bellman equation lacks a boundary condition. The multiplicity of equilibria was first noted in Tsutsui and Mino (1990). In an early application, Dockner and Long (1993) argue that a proper choice of nonlinear strategies can resolve in a non-cooperative way the tragedy of the commons. Less known is the existence of multiple open-loop Nash equilibria, including the possibility of limit cycles, see Wirl et al. (1997). Thus, differential games allow, in nonlinear strategies in linear quadratic games, for an entire family of solutions. However, there is no history dependence if one requires the strategies to be stable.

[7] Several of the previously mentioned "Shallow Lake" papers use a differential game framework – Mäler et al. (2000), Wagener (2003) for instance.

6 Conclusions

This paper has given a state-of-the art review of the conditions under which multiple steady-states can arise in representative agents dynamic optimization models. It has developed a typology of these conditions, clarified the properties of steady-states and thresholds that may arise in the different cases, and presented numerical approaches to study the models' global dynamics and Skiba thresholds. One of its main contributions, furthermore, is to have stressed a commonly ignored fact. Even if there is perfect foresight, no externalities, and strict concavity (the economists' workhorse in insuring uniqueness of optimal solutions), history dependence is possible. Thus, it may be a much more pervasive phenomenon in economics than usually assumed. Even in a very well behaved world, the far future may be very different depending on the current conditions. Since the latter are constantly subject to accidental events, otherwise similar economies need not systematically take the same road.

The results presented here are valid for centrally planned or representative agents economies. Partly, they are also valid for differential games. If there are heterogenous agents, the relationship between individual optimal and aggregate behavior can be different, and possibly more complicated, than described here. While all evidence shows that taking into account agents' heterogeneity increases, if anything, the scope for multiple steady-states, their proper modeling and analysis is the subject matter of future research.

7 References

Altman, M. (2000). A behavioral model of path dependency: the economics of profitable inefficiency and market failure. *Journal of Socio-Economics, 29*, 127-145.

Andvig, J. C. (1991). The economics of corruption: A survey. *Studi economici, 43*, 57-94.

Arthur, W. B. (1989). Competing technologies, increasing returns, and lock-in by historical events. *Economic Journal, 99,* 116-131.

Arthur, W. B. (1994a). Inductive reasoning and bounded rationality. *The American Economic Review (Papers and Proceedings), 84*, 406-411.

Arthur, W. B.(1994b). *Increasing Returns and Path Dependence in the Economy.* Ann Arbor: University of Michigan Press.

Ayong Le Kama, A. D. (2001). Sustainable growth, renewable resources and pollution. *Journal of Economic Dynamics and Control, 25*, 1911-1918.

Azariadis, C. (1996). The economics of poverty traps. Part one: Complete markets. *Journal of Economic Growth 1*, 449-486.

Azariadis, C., & Drazen, A. (1990). Threshold externalities in economic development. *Quarterly Journal of Economics*, *105*(2), 501-526.

Becker, G. S., & Murphy, K. M. (1988). A theory of rational addiction. *Journal of Political Economy, 96*, 675-700.

Benhabib, J., & Perli, R. (1994). Uniqueness and indeterminacy: On the dynamics of endogenous growth. *Journal of Economic Theory*, *63*(1), 113-142.

Benhabib, J., Schmitt-Grohe, S., & Uribe, M. (1998). Monetary policy and multiple equilibria. Economic Research Reports, C. V. Starr Center for Applied Economics, New York University, RR 98-02.

Bernard, A.B., & Durlauf, S.N. (1995). Convergence in International Output. *Journal of Applied Econometrics, 71*, 161-174.

Beyn, W. J., Pampel, T., & Semmler, W. (2001). Dynamic optimisation and Skiba sets in economic examples. *Optimal Control Applications and Methods 22*, 251-280.

Blanchard, O. J. (1983). Debt and current account deficit in Brazil. In: P. A. Armella, R. Dornbusch and M. Obstfeld (eds), *Financial Policies and the World Capital Market: The Problem of Latin American Countries* (pp 187-197). Chicago: University of Chicago Press.

Borisov, V., Feichtinger, G., & Kryazhimskii, A. (2000). Optimal enforcement on a pure seller's market of illicit drugs, *Journal of Optimization Theory and Applications, 106*, 1-22.

Brock, W. A. (1973). Some results on the uniqueness of steady-states in multi-sector models of optimum growth when future utilities are discounted, *International Economic Review*, *14*, 535-559.

Brock, W. A. (1974). Money and growth: The case of long-run perfect foresight. *International Economic Review*, *15*, 750-777.

Brock, W. A. (1983). Pricing, predation and entry barriers in regulated industries. In: D. S. Evans (ed), *Breaking Up Bell* (pp.191-229). Amsterdam: North Holland.

Brock, W. A., & Dechert, D. W. (1985). Dynamic Ramsey pricing. *International Economic Review,* 26, 569-591.

Brock, W. A., & Malliaris, A. G. (1989). *Differential Equations, Stability and Chaos in Dynamical Economics*. Amsterdam, New-York, Oxford, Tokyo: North Holland.

Brock, W. A., & Starrett, D. (1999). Nonconvexities in ecological problems. Mimeo, University of Wisconsin.

Caroll, Ch. D., Overland, J., & Weil, D. N. (2000). Saving and growth with habit formation. *American Economic Review, 90*, 341-355.

Caulkins, J.P. (1993). Local drug markets' response to focused police enforcement. *Operations Research, 41*, 848-863.

Caulkins, J.P., Dworak, M., Feichtinger, G., & Tragler, G. (2000). Price-raising drug enforcement and property crime: A dynamic model. *Journal of Economics, 71*, 227-253.

Chamley, C. (1993). Externalities and dynamics in models of learning or doing. *International Economic Review*, *34*, 583-609.

Cowan R., & Gunby Ph. (1996). Sprayed to death: Path dependence, lock-in and pest control strategies. *Economic Journal, 106*, 521-542.

Dechert, D. W. (1984). Has the Averch-Johnson effect been theoretically justified? *Journal of Economic Dynamics and Control*, *8*, 1-17.

Dechert, D. W., & Brock, W. A. (1999). Lakegame. Mimeo, University of Wisconsin. http://dechert.econ.uh.edu/research/lakegame.pdf.

Dechert, D. W., & Nishimura, K. (1983). Complete characterization of optimal growth paths in an aggregative model with a non-concave production function". *Journal of Economic Theory, 31*, 332-354.

Dockner, E. J. (1985). Local stability analysis in optimal control problems with two state variables. In: G. Feichtinger (ed), *Optimal Control Theory and Economic Analysis 2* (pp 89-103). Amsterdam: North Holland.

Dockner, E. J., & Long, N. V. (1993). International pollution control: Co-operative versus non-co-operative strategies. *Journal of Environmental Economics and Management, 25*, 13-29.

Durlauf, S.N., & Johnson P.A. (1995). Multiple Regimes and Cross-Country Growth Behavior. *Journal of Applied Econometrics, 10*, 365-384.

Faral, E. (1949). "Jean Buridan: Maître és arts de l'Université de Paris". *Histoire littéraire de la France XXXVIII* (p. 462-605). Paris: Imprimerie Nationale.

Feichtinger G., Grienauer, W., & Tragler, G. (2002). Optimal dynamic law enforcement. *European Journal of Operational Research, 141*, 58-69.

Feichtinger G., Hartl, R.F., Kort, P.M., & Wirl, F. (2001). The dynamics of a simple relative adjustment cost framework. *German Economic Review*, 2, 255-268.

Feichtinger, G., Novak, A., & Wirl, F.(1994). Limit cycles in intertemporal adjustment models - theory and applications", *Journal of Economic Dynamics and Control*, *18*, 353-380.

Feichtinger, G., & Tragler, G. (2002). Multiple equilibria in an optimal control model for law enforcement. *Proceedings of the Steklov Institute of Mathematics, 236*, 449-460.

Feichtinger G., & Wirl, F. (2000). Instabilities in concave, dynamic, economic optimization. *Journal of Optimization Theory and Applications, 107*, 277-288

Fujita, M., Krugman P., & Venables A. J. (1999). *The Spatial Economy. Cities, Regions, and International Trade*. Cambridge (Mass): MIT Press.

Gavrila, C., Feichtinger, G., Tragler, G., Hartl, R.F., & Kort, P.M. (2003). Path-dependence in a rational addiction model. Forthcoming in *Mathematical Social Sciences*.

Gould, J. P. (1970). Diffusion processes and optimal advertising policy. In: E. S. Phelps et al. (eds), *Microeconomic Foundations of Employment and Inflation Theory* (pp 338-368). London: Macmillan.

Greiner, A., & Semmler, W. (2002). Monetary policy, non-uniqueness of steady states and hysteresis effects. In: Rheinhard Neck (ed), *Modeling and Control of Economic Systems* (pp. 323-328). Oxford: Elsevier Science, Ltd.

Grüne, L., & Semmler, W. (2002). Using dynamic programming with adaptive grid scheme for optimal control problems in economics. Working paper, Center for Empirical Macroeconomics, University of Bielefeld.

Hartl, R. F., Kort, P.M., Feichtinger, G., & Wirl, F. (2003). Multiple equilibria and thresholds due to relative investment costs: Non-concave - concave, focus - node, continuous - discontinuous. *Journal of Optimization Theory and Applications* forthcoming.

Haunschmied, J. L., Kort, P. M., Hartl, R. F., & Feichtinger, G. (2003). A DNS-curve in a two-state capital accumulation model: A numerical analysis. *Journal of Economic Dynamics and Control, 27*, 701-716.

Hof, F. X., & Wirl, F. (2000). Status in open economy versions of the Ramsey model. Mimeo, University of Vienna.

Kato , M., & Semmler, W. (2001). History dependence and hysteresis effects in an investment model with adjustment costs. Working paper, Center for Empirical Macroeconomics, University of Bielefeld.

Kleiman, M. A. R. (1993). Enforcement swamping: A positive-feedback mechanism in rates of illicit activity. *Mathematical and Computer Modelling, 17*, 65-75.

Kort, P., Feichtinger, G., Hartl, R. F., & Haunschmied, J. L. (1998). Optimal enforcement policies (crackdowns) on a illicit drug market. *Optimal Control Applications and Methods, 19*, 169-184.

Krugman, P. (1991). History versus expectations. *Quarterly Journal of Economics, CVI ,* 651-667.

Krugman, P. (1995). *Development, Geography, and Economic Theory.* Cambridge (Mass.) and London: MIT Press.

Kurz, M. (1968). Optimal economic growth and wealth effects. *International Economic Review, 9*, 348-357.

Ladron-de-Guevara, A., Ortigueira, S., & Santos, M. S. (1999). A two-sector model of endogenous growth with leisure. *Review of Economic Studies, 66*, 609-631.

Lewis, T. R., & Schmalensee, R. (1983). Optimal use of renewable resources with nonconvexities in production. In: L. J. Mirman and D. F. Spulber (eds), *Essays in the Economics of Renewable Resources* (pp 95-111). Amsterdam: North Holland.

Liski M. (2001). Thin versus thick CO2 market. *Journal of Environmental Economics and Management, 41*, 295-311.

Liviatan, N., & Samuelson, P. A. (1969). Notes on turnpikes: Stable and unstable. *Journal of Economic Theory, 1*, 454-475.

Lucas, R. (1988). On the mechanics of endogenous growth. *Journal of Monetary Economics*, *22*, 3-42.

Majumdar, M., Mitra, T., & Nishimura, K. (eds) (2000). *Optimization and Chaos*. Berlin: Springer.

Mäler, K. G. (2000). Development, ecological resources and their management: A study of complex dynamic systems", Joseph Schumpeter Lecture, *European Economic Review, 44*, 645-665.

Mäler, K. G., Xepapadeas, A., & de Zeeuw, A. (2000). The economics of shallow lakes. Preprint, The Beijer International Institute of Ecological Economics, http://www.beijer.kva.se/publications/pdf-archive/disc131.pdf.

Marrewijk, C.V., & Verbeek, J. (1993). On opulence driven poverty traps. J*ournal of Population Economics, 6*, 67-81.

Michael, B. (1985). *Johannes Buridan: Studien zu seinem Leben, seinen Werken und zur Rezeption seiner Theorien im Europa des späten Mittelalters.* Inaugural-Dissertation, 2 vols. Berlin: Freie Universität Berlin.

Orphanides, A., & Zervos, V. (1998). Myopia and addictive behaviour. *Economic Journal*, *108*, 75-91.

Romer, P. (1990). Endogenous technical change. *Journal of Political Economy, 98*(5), 71-101.

Rondeau, D. (2001). Along the way back from the brink. *Journal of Environmental Economics and Management, 42*, 156-182.

Santos, M. S. (2002). On the non-existence of continuous Markov equilibria in competitive-market economies. *Journal of Economic Theory*, *105*, 73-98.

Schelling, T. S. (1967). Economics and criminal enterprise. *The Public Interest, 7*, 61-78.

Schelling, T. S. (1973). Hockey helmets, concealed weapons, and daylight saving: a study of binary choices with externalities. *Journal of Conflict Resolution, 17*, 381-428.

Semmler, W., & Sieveking, M. (1999). Using vector field analysis for studying debt dynamics. Paper prepared for the North American Winter Meeting of the Econometric Society, Chicago. Mimeo, University of Bielefeld.

Sieveking, M., & Semmler, W. (1997). The present value of resources with large discount rates. *Applied Mathematics and Optimisation: An International Journal*, *35*, 283-309.

Skiba, A. K. (1978). Optimal growth with a convex-concave production function. *Econometrica, 46*, 527-539.

Tahvonen, O., & Salo, S. (1996). Nonconvexities in optimal pollution accumulation. *Journal of Environmental Economics and Management, 31*, 160-177.

Tahvonen, O., & Withagen, C. (1996). Optimality of irreversible pollution accumulation. *Journal of Economic Dynamics and Control, 20*, 1775-1795.

Tragler, G., Caulkins, J.P., & Feichtinger, G. (2001). Optimal dynamic allocation of treatment and enforcement in illicit drug control. *Operations Research, 49*, 352-362.

Tsutsui, S., & Mino, K. (1990). Non-linear strategies in dynamic competition with sticky prices. *Journal of Economic Theory, 52*, 136-161.

Wagener, F. O. O. (2003). Skiba points and heteroclinic bifurcations, with applications to the shallow lake system. *Journal of Economic Dynamics and Control, 27,* 1533-1561.

Wirl, F. (1994). The Ramsey model revisited - the optimality of cyclical consumption and growth. *Journal of Economics, 60*, 81-98.

Wirl, F. (2003). Sustainable growth, renewable resources and pollution: Thresholds and cycles. *Journal of Economic Dynamics and Control,* forthcoming.

Wirl, F., Feichtinger, G., Dawid, H., & Novak, A. (1997). Corruption in democratic systems: A differential game between politicians and the press. In: H. G. Natke and Y. Ben-Haim (eds), *Uncertainty: Models and Measures* (pp 124-136). Berlin: Akademie Verlag.

Xie, D. (1994). Divergence in economic performance: Transitional dynamics with multiple equilibria. *Journal of Economic Theory, 63*, 97-112.